

# On Pocrims and Hoops

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October 17, 2014

## Abstract

Pocrims and suitable specialisations thereof are structures that provide the natural algebraic semantics for a minimal affine logic and its extensions. Hoops comprise a special class of pocrims that provide algebraic semantics for what we view as an intuitionistic analogue of the classical multi-valued Łukasiewicz logic. We present some contributions to the theory of these algebraic structures. We give a new proof that the class of hoops is a variety. We use a new indirect method to establish several important identities in the theory of hoops: in particular, we prove that the double negation mapping in a hoop is a homomorphism. This leads to an investigation of algebraic analogues of the various double negation translations that are well-known from proof theory. We give an algebraic framework for studying the semantics of double negation translations and use it to prove new results about the applicability of the double negation translations due to Gentzen and Glivenko.

## 1 Introduction

Pocrims provide the natural algebraic models for a minimal affine logic,  $\mathbf{AL}_m$ , while hoops provide the models for what we view as a minimal analogue,  $\mathbf{LL}_m$ , of Łukasiewicz's classical infinite-valued logic  $\mathbf{LL}_c$ . This paper presents some new results on the algebraic structure of pocrims and hoops. Our main motivation for this work is in the logical aspects: we are interested in criteria for provability in  $\mathbf{AL}_m$ ,  $\mathbf{LL}_m$  and related logics. We develop a useful practical test for provability in  $\mathbf{LL}_m$  and apply it to a range of problems including a study of the various double negation translations in these logics.

We begin in Section 2 with a brief introduction to the logical background and then give the definitions and basic theory of the algebraic structures. Since we believe the algebraic approach will be unfamiliar to some readers

who share our interest in the logical issues, this part of the paper is largely expository, bringing together material that is scattered over the literature. We illustrate the material with a number of examples, including all pocrimms of order 2, 3 and 4. Some of our later results depend on the existence of finite pocrimms satisfying or failing to satisfy certain identities: the witnesses were all found using the late Bill McCune’s Mace4 program [20], which has proved an invaluable tool in our work.

In Section 3 we review the algebraic semantics for the logics introduced in Section 2 and prove the soundness and completeness of pocrimms and appropriate specialisations thereof to these logics. Again this section is largely expository however it concludes, with a new proof that the class of hoops is a variety. The proof provides an algorithm for translating a proof tree in the logic  $\mathbf{LL}_m$  into a semantically equivalent equational proof.

The equational theory of hoops is known to be decidable and it follows from work of Bova and Montagna [7] that the decision problem is in PSPACE. Unfortunately, their decision procedure is infeasible in practice, even on small examples. In Section 4, we attempt to mitigate this difficulty. We begin by reviewing known results on the equational decision problem for involutive hoops (i.e., hoops that satisfy an algebraic analogue of the law of double negation elimination). The variety of involutive hoops can be shown to be definitionally equivalent to the well-known variety of MV-algebras and the equational theory of MV-algebras reduces to the theory of linear real arithmetic. We then reduce the decision problem for an identity in a general hoop to restricted classes of finitely generated hoops enjoying special algebraic properties. This falls short of a decision procedure, but provides an efficient heuristic that can be used to prove many important identities, whose formal proofs, if known, are extremely intricate. We give several interesting applications of this method, e.g., we show that the set of idempotent elements in a hoop is the universe of a subhoop.

In Section 5, we use the method of Section 4 to show that the double negation mapping in a hoop is a homomorphism. We undertake an algebraic investigation of the double negation translations of Kolmogorov, Gentzen and Glivenko. Kolmogorov’s translation is shown to be correct for any extension of affine logic. The Gentzen and Glivenko translations are correct for intuitionistic Łukasiewicz logic, but there are weaker extensions of affine logic for which Gentzen is correct but Glivenko is not and *vice versa*.

(Comp)	$(A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C)$
(Comm)	$A \otimes B \multimap B \otimes A$
(Curry)	$(A \otimes B \multimap C) \multimap (A \multimap B \multimap C)$
(Uncurry)	$(A \multimap B \multimap C) \multimap (A \otimes B \multimap C)$
(Wk)	$A \otimes B \multimap A$
(EFQ)	$1 \multimap A$
(DNE)	$A^{\perp\perp} \multimap A$
(CWC)	$A \otimes (A \multimap B) \multimap B \otimes (B \multimap A)$
(Con)	$A \multimap A \otimes A$

Figure 1: Axiom Schemata

## 2 Background

While the main emphasis of this paper is on algebraic structures, our main motivation for studying those structures stems from an interest in certain substructural propositional logics. We now define those logics.

### 2.1 Nine Logics

We work in a language,  $\mathcal{L}$ , built from a countable set of variables  $\text{Var} = \{V_1, V_2, \dots\}$ , the constant 1 (falsehood) and the binary connectives  $\multimap$  (implication) and  $\otimes$  (conjunction). We write  $A^\perp$  for  $A \multimap 1$  and 0 for  $1 \multimap 1$ . Our choice of notation for connectives is that commonly used for affine logic, since all the systems we consider are extensions of intuitionistic affine logic. Our use of 1 rather than 0 for falsehood is taken from continuous logic [3], which motivated our work in this area.

As usual, we adopt the convention that  $\multimap$  associates to the right and has lower precedence than  $\otimes$ , which in turn has lower precedence than  $(\cdot)^\perp$ . So, for example, the brackets in  $(A \otimes (B^\perp)) \multimap (C \multimap (D \otimes F))$  are all redundant, while those in  $((A \multimap B) \multimap C) \otimes D)^\perp$  are all required.

If  $T$  is a set of formulas in the language  $\mathcal{L}$ , the *deductive closure*,  $\overline{T}$ , of  $T$  is the smallest subset of  $\mathcal{L}$  that contains  $T$  and is closed under *modus ponens* (i.e., if  $A \in \overline{T}$  and  $A \multimap B \in \overline{T}$  then  $B \in \overline{T}$ ). If  $T = \overline{T}$ , we say  $T$  is *deductively closed* or a *theory*. For our purposes in this paper a logic is just a theory. However, we will often write “ $T$  proves  $A$ ” or “ $A$  is derivable in  $T$ ” as a suggestive alternative to  $A \in T$ . If  $S$  and  $T$  are sets of formulas, e.g., theories or axiom schemata, we write  $S + T$  for  $\overline{S \cup T}$ .

<b>AL<sub>m</sub></b>	(Comp) + (Comm) + (Curry) + (Uncurry) + (Wk)
<b>AL<sub>i</sub></b>	<b>AL<sub>m</sub></b> + (EFQ)
<b>AL<sub>c</sub></b>	<b>AL<sub>i</sub></b> + (DNE)
<b>LL<sub>m</sub></b>	<b>AL<sub>m</sub></b> + (CWC)
<b>LL<sub>i</sub></b>	<b>LL<sub>m</sub></b> + (EFQ)
<b>LL<sub>c</sub></b>	<b>LL<sub>i</sub></b> + (DNE)
<b>ML</b>	<b>AL<sub>m</sub></b> + (Con)
<b>IL</b>	<b>ML</b> + (EFQ)
<b>BL</b>	<b>IL</b> + (DNE)

Figure 2: Logics

We will consider nine axiom schemata as shown in the table of Figure 1. These are: *composition*, *commutativity of conjunction*, *currying*, *uncurrying*, *weakening*, *ex falso quodlibet*, *double negation elimination*, *commutativity of weak conjunction and contraction*.

We then consider nine combinations of these axiom schema, as shown in Figure 3. **AL<sub>m</sub>**, **AL<sub>i</sub>**, **AL<sub>c</sub>**, **LL<sub>m</sub>**, **LL<sub>i</sub>** and **LL<sub>c</sub>** are minimal, intuitionistic and classical variants of affine logic and Łukasiewicz logic. **ML**, **IL** and **BL** have both weakening, (Wk), and contraction, (Con), and so are the implication-conjunction fragments of the usual minimal, intuitionistic and boolean logics. Over **AL<sub>m</sub>**, the schema (Con) implies the schema (CWC). In fact, as discussed in [2], one can interpret (CWC) as a weak form of the contraction rule. We can consequently depict our nine logics in the 2-dimensional diagram shown in Figure 3 (in which the rectangles are push-outs in the poset of deductively closed subsets of  $\mathcal{L}$ ).

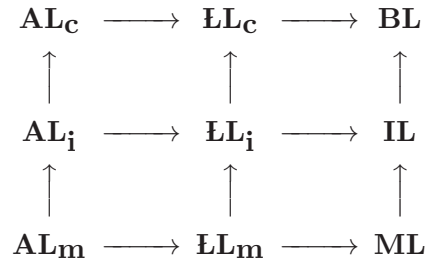


Figure 3: Relationships between the Logics

It was shown in the 1950s by Rose and Rosser [22] (and also using a different method of proof by Chang [11]) that the Hilbert-style system  $\mathbf{L}$  with the following axiom schemata<sup>1</sup> is sound and complete for Łukasiewicz's many-valued logical system where the truth values are real numbers in the interval  $[0, 1]$  and where  $\multimap$  and  $(\cdot)^\perp$  are modelled by  $(a, b) \mapsto \min(a + b, 1)$  and  $a \mapsto 1 - a$  respectively<sup>2</sup>.

- (A1)  $A \multimap (B \multimap A)$
- (A2)  $(A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C)$
- (A3)  $((A \multimap B) \multimap B) \multimap ((B \multimap A) \multimap A)$
- (A4)  $(A^\perp \multimap B^\perp) \multimap (B \multimap A)$

Note that in  $\mathbf{L}$  conjunction can be defined as  $A \otimes B := (A^\perp \multimap B)^\perp$ . We will see in Section 4.1 that our  $\mathbf{LL}_C$  is equivalent to  $\mathbf{L}$ .

## 2.2 Pocrims and Hoops

**Definition 2.2.1** A pocrim<sup>3</sup>  $\mathbf{P}$  is a structure for the signature  $(0, +, \rightarrow)$  of type  $(0, 2, 2)$  satisfying the following laws, in which  $x \geq y$  is an abbreviation for  $x \rightarrow y = 0$ :

- $(x + y) + z = x + (y + z)$  [m<sub>1</sub>]
- $x + y = y + x$  [m<sub>2</sub>]
- $x + 0 = x$  [m<sub>3</sub>]
- $x \geq x$  [o<sub>1</sub>]
- if  $x \geq y$  and  $y \geq z$ , then  $x \geq z$  [o<sub>2</sub>]
- if  $x \geq y$  and  $y \geq x$ , then  $x = y$  [o<sub>3</sub>]
- if  $x \geq y$ , then  $x + z \geq y + z$  [o<sub>4</sub>]
- $x \geq 0$  [b]
- $x + y \geq z$  iff  $x \geq y \rightarrow z$ . [r]

We will see that pocrimms provide models for our logics:  $\rightarrow$  is the semantic counterpart of the syntactic implication  $\multimap$ , whereas  $+$  corresponds to the

<sup>1</sup> Following Łukasiewicz, Rose and Rosser used Polish notation. Rose and Rosser write  $CAB$  for our  $A \multimap B$  and  $NA$  for our  $A^\perp$ . Chang followed this in the relatively few fragments of syntax that appear in his treatment.

<sup>2</sup> Throughout this paper we adopt the convention that truth values are ordered by increasing logical strength, so 0 represents truth and 1 represents falsehood.

<sup>3</sup>The name is an acronym for “partially ordered, commutative, residuated, integral monoid”. Strictly speaking, this is a *dual* pocrim, since we order it by increasing logical strength and write it additively.

syntactic conjunction  $\otimes$ . As with the syntactic connectives, we adopt the convention that  $\rightarrow$  associates to the right and has lower precedence than  $+$ . So the brackets in  $x + (x \rightarrow y)$  are necessary while those in  $x \rightarrow (y \rightarrow z)$  may be omitted. Throughout this paper, we adopt the convention that if  $\mathbf{P}$  is a structure then  $P$  is its universe.

The laws  $[\mathbf{m}_i]$ ,  $[\mathbf{o}_j]$  and  $[\mathbf{b}]$  say that  $(P; 0, +; \geq)$  is a partially ordered commutative monoid with the identity 0 as least element. Law  $[\mathbf{r}]$ , the *residuation property*, says that for any  $y$  and  $z$  the set  $\{x \mid x + y \geq z\}$  is non-empty and has  $y \rightarrow z$  as least element. Taking  $x = y \rightarrow z$  in  $[\mathbf{r}]$  and using  $[\mathbf{o}_1]$ , we have that  $(y \rightarrow z) + y \geq z$ , an algebraic analogue of *modus ponens*.

A pocrim is said to be *bounded* if it has a (necessarily unique) *annihilator*, i.e., an element 1 such that for every  $x$  we have:

$$1 = x + 1. \quad [\mathbf{ann}]$$

In a bounded pocrim  $\mathbf{P}$ , we have that  $1 = x + 1 \geq x + 0 = x$  for any  $x$ , so that  $(M; \geq)$  is indeed a bounded ordered set. We write  $\neg x$  for  $x \rightarrow 1$  (and give  $\neg$  higher precedence than the binary operators). Note that any finite pocrim  $\mathbf{P}$  is bounded, the annihilator being given by  $\sum_{x \in P} x$ .

A pocrim is said to be *involutive* if it is bounded and satisfies the double-negation identity:

$$\neg \neg x = x. \quad [\mathbf{dne}]$$

We will often write  $\delta(x)$  for  $\neg \neg x$ . In any bounded pocrim, the set  $\{0, 1\}$  is closed under  $+$  and  $\rightarrow$  and so, as  $\neg 0 = 1$  and  $\neg 1 = 0$ ,  $\{0, 1\}$  is the universe of an involutive subpocrim.

**Example 2.2.1** *There is a unique pocrim  $\mathbb{B}$  with two elements. It is involutive and provides the standard model for classical Boolean logic.*

If  $x$  and  $y$  are elements of a pocrim,  $x + (x \rightarrow y)$  is an upper bound for  $x$  and  $y$  as is  $y + (y \rightarrow x)$ . Logically, we can view either of these as a weak form of conjunction. Pocrims in which the two upper bounds coincide turn out to have many pleasant properties, motivating the following definition.

**Definition 2.2.2 (Büchi & Owens[8])** *A hoop<sup>4</sup> is a pocrim that satisfies commutativity of weak conjunction:*

$$x + (x \rightarrow y) = y + (y \rightarrow x). \quad [\mathbf{cwc}]$$

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<sup>4</sup> Büchi and Owens [8] write of hoops that “their importance ... merits recognition with a more euphonious name than the merely descriptive “commutative complemented monoid””. Presumably they chose “hoop” as a euphonious companion to “group” and “loop”.

The following lemma provides some useful characterisations of hoops.

**Lemma 2.2.1** *If  $\mathbf{P}$  is a pocrim, the following are equivalent:*

1.  $\mathbf{P}$  is a hoop. I.e.,  $\mathbf{P}$  satisfies  $x + (x \rightarrow y) = y + (y \rightarrow x)$ .
2.  $\mathbf{P}$  is naturally ordered. I.e., for every  $x, y \in P$  such that  $x \geq y$ , there is  $z \in P$  such that  $x = y + z$ .
3. For every  $x, y \in P$  such that  $x \geq y$ ,  $x = y + (y \rightarrow x)$ .
4.  $\mathbf{P}$  satisfies  $x + (x \rightarrow y) \geq y + (y \rightarrow x)$

**Proof:**  $1 \Rightarrow 2$ : Assume that  $\mathbf{P}$  satisfies  $x + (x \rightarrow y) = y + (y \rightarrow x)$  and that  $x, y \in P$  satisfy  $x \geq y$ , i.e.,  $x \rightarrow y = 0$ . Taking  $z = y \rightarrow x$ , we have  $x = x + 0 = x + (x \rightarrow y) = y + (y \rightarrow x) = y + z$ .

$2 \Rightarrow 3$ : Assume that  $\mathbf{P}$  is naturally ordered and that  $x, y \in P$  satisfy  $x \geq y$ . Then  $x = y + z$  for some  $z$ . By the residuation property, we have  $z \geq y \rightarrow x$ , hence  $x = y + z \geq y + (y \rightarrow x) \geq x$  and so  $x = y + (y \rightarrow x)$ .

$3 \Rightarrow 4$ : assume that  $\mathbf{P}$  satisfies  $x = y + (y \rightarrow x)$  whenever  $x, y \in P$  and  $x \geq y$ . Given any  $x, y \in P$ , we have  $x + (x \rightarrow y) \geq y$ , whence  $x + (x \rightarrow y) = y + (y \rightarrow x + (x \rightarrow y)) \geq y + (y \rightarrow x)$ .

$4 \Rightarrow 1$ : exchange  $x$  and  $y$  and use the fact that  $\geq$  is antisymmetric.  $\blacksquare$

We will now give an outline of some basic algebraic properties of pocrim and hoops omitting most of the proofs. See [21] for further information about pocrim in general and involutive pocrim in particular. See [5] for further information about hoops.

Given a linearly ordered abelian group  $\mathbf{G}$ , there is a hoop  $\mathbf{G}^{\geq 0} = (\{x : G \mid x \geq 0\}; 0, +, \rightarrow)$ , where  $x \rightarrow y = \max(0, y - x)$ . So for example, taking  $G$  to be the additive group of real numbers, we have the hoop  $\mathbb{R}^{\geq 0}$  whose elements are non-negative real numbers. Given an element  $a$  of a linearly ordered hoop  $\mathbf{H}$ , there is a bounded hoop  $\mathbf{H}^{\leq a} = (\{x : H \mid x \leq a\}; 0, +_a, \rightarrow)$  where  $x +_a y = \min(a, x + y)$  so that  $a$  becomes the annihilator. If we compose these two constructions, the resulting bounded hoop  $\mathbf{G}^{[0, a]}$  is involutive, since it satisfies  $\neg x = a - x$ .

**Example 2.2.2** *We write  $[0, 1]$  for the involutive hoop  $\mathbb{R}^{[0, 1]}$  obtained by the above constructions taking  $\mathbf{G} = (\mathbb{R}; 0, +)$  and  $a = 1$ . Thus the universe of  $[0, 1]$  is the unit interval and the operations are given by:*

$$x + y = \min(x + y, 1) \qquad x \rightarrow y = \max(y - x, 0)$$

(where we write  $\dot{+}$  rather than  $+$  for the hoop operation to distinguish it from addition of real numbers).  $[\mathbf{0}, \mathbf{1}]$  provides an infinite model of classical Lukasiewicz logic  $\mathbf{LL}_{\mathbf{C}}$  (as does  $\mathbf{G}^{[0,1]}$  for any dense subgroup of  $(\mathbb{R}; 0, +)$  containing 1).

**Example 2.2.3** For any integer  $m \geq 1$ , define  $\mathbf{R}_m$  to be the additive subgroup of  $\mathbb{Q}$  generated by  $\frac{1}{m}$ . For  $n \geq 2$ , let  $\mathbf{L}_n = \mathbf{R}_{n-1}^{[0,1]}$ . Thus the universe of  $\mathbf{L}_n$  is  $L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$  and the operations  $+$  and  $\rightarrow$  on  $L_n$  are given by the same formulas as for  $[\mathbf{0}, \mathbf{1}]$  in Example 2.2.2. The hoops  $\mathbf{L}_n$  are involutive and provide natural finite models of classical Lukasiewicz logic  $\mathbf{LL}_{\mathbf{C}}$ .

A hoop  $\mathbf{H}$  is said to be a *Wajsberg hoop* if it satisfies  $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ . This is the algebraic equivalent of the axiom schema (A3) of Section 2.1. A bounded hoop is Wajsberg iff it is involutive. There are, however, unbounded Wajsberg hoops, for instance:

**Example 2.2.4** The unbounded hoop  $\mathbb{R}^{\geq 0}$  is Wajsberg. In fact, in  $\mathbb{R}^{\geq 0}$   $(x \rightarrow y) \rightarrow y$  and  $(y \rightarrow x) \rightarrow x$  are both equal to  $\min(x, y)$ .

If  $\mathbf{C}$  and  $\mathbf{D}$  are pocrim, the *ordinal sum*,  $\mathbf{C} \frown \mathbf{D}$ , is the pocrim  $(C \sqcup (D \setminus \{0\}), 0, +, \rightarrow)$  where  $+$  and  $\rightarrow$  extend the given operations on  $C$  and  $D$  to the disjoint union  $C \sqcup (D \setminus \{0\})$  in such a way that whenever  $c \in C$  and  $0 \neq d \in D$ ,  $c + d = d$  (implying that  $c \rightarrow d = d$  and  $d \rightarrow c = 0$ ). Thus the order type of  $\mathbf{C} \frown \mathbf{D}$  is the concatenation of the partial orders  $(C; \geq)$  and  $(D \setminus \{0\}; \geq)$ . If  $D \neq \{0\}$ ,  $\mathbf{C} \frown \mathbf{D}$  is bounded iff  $\mathbf{D}$  is bounded and can only be involutive if  $C = \{0\}$ , since if  $0 \neq c \in C$ , then, in  $\mathbf{C} \frown \mathbf{D}$ , we have  $\neg c = 1$ , so that  $\neg \neg c = 0 \neq c$ .  $\mathbf{C} \frown \mathbf{D}$  is a hoop iff both  $\mathbf{C}$  and  $\mathbf{D}$  are hoops.

**Example 2.2.5** Apart from  $\mathbf{L}_3$  there is one other pocrim with 3 elements, namely  $\mathbf{G}_3 = \mathbb{B} \frown \mathbb{B}$ .  $\mathbf{G}_3$  is the first non-Boolean example in the sequence of idempotent pocrim defined by the equations  $\mathbf{G}_2 = \mathbb{B}$  and  $\mathbf{G}_{n+1} = \mathbf{G}_n \frown \mathbb{B}$ .  $\mathbf{G}_n = \{0, x_1, x_2, \dots, x_{n-2}, 1\}$  with  $0 < x_1 < x_2 < \dots < x_{n-2} < 1$  and with operations defined by

$$x + y = \max\{x, y\} \quad x \rightarrow y = \begin{cases} y & \text{if } y > x \\ 0 & \text{otherwise} \end{cases}$$

The  $\mathbf{G}_n$  are finite models of intuitionistic propositional logic  $\mathbf{IL}$ . They were used by Gödel to prove that  $\mathbf{IL}$  requires infinitely many truth values [16]. In  $\mathbf{G}_n$ ,  $\neg x = 1$  unless  $x = 1$ , so for  $n > 2$ ,  $\mathbf{G}_n$  is not involutive.



**Example 2.2.6** It can be shown that there are 7 pocrimms with 4 elements:  $\mathbb{B} \times \mathbb{B}$ ,  $\mathbf{L}_4$ ,  $\mathbf{G}_4$ ,  $\mathbb{B} \cap \mathbf{L}_3$ ,  $\mathbf{L}_3 \cap \mathbb{B}$ ,  $\mathbf{P}_4$  and  $\mathbf{Q}_4$ .  $\mathbf{P}_4$  and  $\mathbf{Q}_4$  are the smallest pocrimms that are not hoops and are as follows:  
 $\mathbf{P}_4$  comprises the chain  $0 < p < q < 1$ . The operation tables for  $\mathbf{P}_4$  are as follows.

$+$	0	$p$	$q$	1	$\rightarrow$	0	$p$	$q$	1	$\delta$
0	0	$p$	$q$	1	0	0	$p$	$q$	1	0
$p$	$p$	1	1	1	$p$	0	0	$p$	$p$	$p$
$q$	$q$	1	1	1	$q$	0	0	0	$p$	$q$
1	1	1	1	1	1	0	0	0	0	1

(where for future reference we also tabulate the double negation mapping,  $\delta$ ). In  $\mathbf{P}_4$ ,  $\delta(q) = p$ , so  $\mathbf{P}_4$  is not involutive. Moreover  $\mathbf{P}_4$  is not a hoop since it is not naturally ordered: there is no  $z$  with  $p + z = q$ . However, the image of double negation is a subpocrim with universe  $\{0, p, 1\}$  isomorphic to the involutive hoop  $\mathbf{L}_3$ .

$\mathbf{Q}_4$  comprises the chain  $0 < u < v < 1$  and has operation tables as follows:

$+$	0	$u$	$v$	1	$\rightarrow$	0	$u$	$v$	1	$\delta$
0	0	$u$	$v$	1	0	0	$u$	$v$	1	0
$u$	$u$	$u$	1	1	$u$	0	0	$v$	$v$	$u$
$v$	$v$	1	1	1	$v$	0	0	0	$u$	$v$
1	1	1	1	1	1	0	0	0	0	1

Like  $\mathbf{P}_4$ ,  $\mathbf{Q}_4$  is not naturally ordered and hence not a hoop, because there is no  $z$  with  $u + z = v$ .  $\mathbf{Q}_4$  is involutive.

An *ideal* in a hoop  $\mathbf{H}$  is a subset that forms the universe of a downwards closed subhoop. Trivially  $H$  itself and  $\{0\}$  are ideals. If  $X \subseteq H$ , the ideal generated by  $X$  comprises the set of all  $y \in H$  such that  $y \leq x_1 + \dots + x_k$  for some list  $x_1, \dots, x_k$  of elements of  $X$ .

If  $f : \mathbf{H} \rightarrow \mathbf{K}$  is a hoop homomorphism, we define  $\ker(f)$ , the *kernel* of  $f$ , by  $\ker(f) = \{x : H \mid f(x) = 0\}$ . It is easy to verify that  $\ker(f)$  is an ideal. Conversely, given an ideal  $I$  in  $\mathbf{H}$ , the relation  $\theta$  on  $H$  defined by  $x \theta y$  iff  $(x \rightarrow y) + (y \rightarrow x) \in I$  defines a congruence on  $\mathbf{H}$  such that if  $p : \mathbf{H} \rightarrow \mathbf{H}/\theta$  is the natural projection of  $\mathbf{H}$  onto the quotient hoop<sup>5</sup>  $\mathbf{H}/\theta$ , then  $\ker(p) = I$ . This gives an isomorphism between the lattice of congruences on  $\mathbf{H}$  and its lattice of ideals. We write  $\mathbf{H}/I$  for the quotient of  $\mathbf{H}$  by the congruence corresponding to the ideal  $I$ .

<sup>5</sup> We shall show in Section 3.2.2 that the class of hoops is a variety, so the quotient of a hoop by a congruence is in fact a hoop.

**Example 2.2.7** If  $\mathbf{C}$  and  $\mathbf{D}$  are hoops,  $C$  is an ideal in the ordinal sum  $\mathbf{C} \smallfrown \mathbf{D}$  and the quotient  $(\mathbf{C} \smallfrown \mathbf{D})/C$  is isomorphic to  $\mathbf{D}$  via an isomorphism that is left inverse to the natural inclusion of  $\mathbf{D}$  in  $\mathbf{C} \smallfrown \mathbf{D}$ .

If  $\mathbf{P}$  is a pocrim,  $n \in \mathbb{N}$  and  $x \in P$ , we write  $nx$  for the sum  $\sum_{i=1}^n x$ .  $\mathbf{P}$  is said to be *archimidean* if whenever  $x, y \in \mathbf{P} \setminus \{0\}$ , there is  $n \in \mathbb{N}$  such that  $nx \geq y$ . By the equivalence between ideals and congruences, a hoop  $\mathbf{H}$  is simple, i.e., admits no non-trivial congruences, iff the ideal generated by any non-zero element of  $H$  is  $H$  itself. It follows that a hoop is simple iff it is archimedean. So, for example, the  $\mathbf{L}_n$  are all simple, while  $\mathbf{G}_n$  is simple iff  $n = 2$ .

**Example 2.2.8** Let  $\mathbf{E}$  be the plane  $\mathbb{R} \times \mathbb{R}$  given the structure of a linearly ordered abelian group under vector addition and the lexicographic ordering and let  $a = (1, 0)$ .  $\mathbf{E}^{[0, a]}$  is not archimedean: the elements of  $E^{[0, a]}$  on the  $y$ -axis form a subhoop,  $\mathbf{Y}$ , such that  $ny < a$  for any  $n \in \mathbb{N}$  and  $y \in Y$ . The projection  $\pi_1 : E \rightarrow \mathbb{R}$  onto the  $x$ -axis induces a surjective hoop homomorphism  $f : \mathbf{E}^{[0, a]} \rightarrow [\mathbf{0}, \mathbf{1}]$  and  $\ker(f) = Y$ .  $[\mathbf{0}, \mathbf{1}]$  and  $\mathbf{Y}$  are clearly archimedean, and hence simple. It follows that  $Y$  is the only non-trivial ideal in  $\mathbf{E}^{[0, a]}$ .

Recall that an algebra  $\mathbf{A}$  is said to be *subdirectly irreducible* if the intersection  $\Psi = \bigcap (\text{Con } \mathbf{A} \setminus \Delta)$  of all its congruences other than the identity congruence,  $\Delta$ , is not equal to  $\Delta$ . Using the correspondence between congruences and ideals,  $\mathbf{E}^{[0, a]}$  in the above example may be seen to be subdirectly irreducible, as its only ideals are  $\{0\}$ ,  $Y$  and  $E^{[0, a]}$ .

### 3 Algebraic Semantics

In this section, we begin by rendering the Hilbert-style systems of Section 2.1 more tractable by studying the derivability relation in  $\mathbf{AL}_m$  and its extensions. We then give the semantics for the language  $\mathcal{L}$  in a pocrim and show that the logics of Figure 3 are each sound and complete for a corresponding class of pocrim. We use the semantics to give a new proof that hoops form a variety.

#### 3.1 Derivability

If  $T$  is any subset of  $\mathcal{L}$ , we say  $B$  is *derivable* from  $A$  in  $T$  and write  $A \geq_T B$ , if  $A \multimap B$  is provable in  $T$ . We say  $A$  and  $B$  are *equivalent* in  $T$  and write  $A \simeq_T B$ .

$B$ , if  $A \geq_T B$  and  $B \geq_T A$ . Thus  $A \geq_{\mathbf{AL}_m} B$  means that  $A \multimap B$  can be derived from the axiom schemata (Comp), (Comm), (Curry), (Uncurry) and (Wk) using *modus ponens*. When the  $T$  in question is clear from the context we just write  $\geq$  and  $\simeq$ . Our goal is to find properties of these relations that make it easy to prove facts such as  $A \multimap B \multimap D \otimes C \simeq_T B \otimes A \multimap C \otimes D$ , where  $T$  extends  $\mathbf{AL}_m$ .

**Lemma 3.1.1** *Let the theory  $T$  extend  $\mathbf{AL}_m$ . Then  $\geq_T$  is a pre-order and  $\simeq_T$  is an equivalence relation.*

**Proof:** Recall that a pre-order is a transitive and reflexive relation. By definition, if  $A \geq B$  and  $B \geq C$ , then  $A \multimap B$  and  $B \multimap C$  are both derivable in  $\mathbf{AL}_m$  and then using axiom (Comp) and two applications of *modus ponens*, we can derive  $A \multimap C$ , so that  $A \geq C$ . So  $\geq$  is transitive. Now let  $D$  be any provable formula, say the instance  $V_1 \otimes V_2 \multimap V_1$  of (Wk). We can then derive  $A \multimap A$  as follows:

- |    |   |              |
|----|---|--------------|
| 1: | $D$   | [(Wk)]       |
| 2: | $A \otimes D \multimap A$                                       | [(Wk)]       |
| 3: | $D \otimes A \multimap A \otimes D$                             | [(Comm)]     |
| 4: | $(A \otimes D \multimap A) \multimap (D \otimes A \multimap A)$ | [3, (Comp)]  |
| 5: | $D \otimes A \multimap A$                                       | [2, 4]       |
| 6: | $D \multimap A \multimap A$                                     | [5, (Curry)] |
| 7: | $A \multimap A$   | [1, 6]       |

(Here a justification such as [3, (Comp)] indicates an application of *modus ponens* with the result of line 3 as the cut-formula and an instance of (Comp) as the implication.) So  $\geq$  is reflexive and hence is indeed a pre-order. That  $\simeq$  is an equivalence relation follows immediately. ■

**Lemma 3.1.2** *Let the theory  $T$  extend  $\mathbf{AL}_m$ . For any formula  $A$ , the following are equivalent: (i)  $A$  is provable in  $T$ ; (ii)  $B \geq_T A$  for every formula  $B$ ; (iii)  $B \geq_T A$  for some formula  $B$  that is provable in  $T$ .*

**Proof:** (i)  $\Rightarrow$  (ii): If  $A$  is provable and  $B$  is any formula, then we can derive  $B \multimap A$  as follows. By assumption we have  $A$ . By (Wk) we have  $A \otimes B \multimap A$ , which by (Curry) gives us  $A \multimap B \multimap A$ . Finally, from  $A$  and the  $A \multimap B \multimap A$  we obtain  $B \multimap A$ .

(ii)  $\Rightarrow$  (iii): This is trivial given that provable formulas exist.

(iii)  $\Rightarrow$  (i): By definition, if  $B \geq A$ , then  $B \multimap A$  is provable, so if  $B$  is

provable, then  $A$  follows with one application of *modus ponens*.  $\blacksquare$

In the sequel, as in the following proof, we will often tacitly apply Lemma 3.1.2, typically taking the provable formula  $B$  in part (iii) to be  $0 \equiv 1 \multimap 1$  which is provable by dint of Lemma 3.1.1.

**Lemma 3.1.3** *Let the theory  $T$  extend  $\mathbf{AL}_m$ . With respect to the pre-order  $\geq_T$ ,  $\multimap$  is antimonotonic in its first argument and monotonic in its second argument, while  $\otimes$  is monotonic in both arguments. I.e., for any formulas  $A$ ,  $B$  and  $C$  such that  $A \geq_T B$ , the following hold:*

$$B \multimap C \geq_T A \multimap C \quad (i)$$

$$C \multimap A \geq_T C \multimap B \quad (ii)$$

$$A \otimes C \geq_T B \otimes C \quad (iii)$$

$$C \otimes A \geq_T C \otimes B. \quad (iv)$$

The equivalence relation  $\simeq_T$  is a congruence with respect to both  $\multimap$  and  $\otimes$ . I.e., for any formulas  $A$ ,  $B$  and  $C$  such that  $A \simeq_T B$ , the following hold:

$$B \multimap C \simeq_T A \multimap C \quad (v)$$

$$C \multimap A \simeq_T C \multimap B \quad (vi)$$

$$A \otimes C \simeq_T B \otimes C \quad (vii)$$

$$C \otimes A \simeq_T C \otimes B. \quad (viii)$$

**Proof:** Assume that  $A \geq B$ , i.e., that  $A \multimap B$  is provable in  $T$ . Using *modus ponens* and (Comp), we can derive  $(B \multimap C) \multimap (A \multimap C)$ . So (i) holds. Using (Comm), (Curry), (Uncurry) and (i), we have  $(X \multimap Y \multimap Z) \geq (Y \multimap X \multimap Z)$ . Instantiating  $X$ ,  $Y$  and  $Z$  to  $C \multimap A$ ,  $A \multimap B$  and  $C \multimap B$  respectively, the left-hand side of this inequality becomes an instance of (Comp) and hence the right-hand side is provable in  $T$ . But the right-hand side is exactly what we need to derive  $(C \multimap A) \multimap (C \multimap B)$  from our assumption  $A \multimap B$  using *modus ponens*. So (ii) holds. We now have the following inequalities:

$$0 \geq B \otimes C \multimap B \otimes C \quad (\text{Lemma 3.1.1})$$

$$\geq B \multimap C \multimap B \otimes C \quad (\text{Curry})$$

$$\geq A \multimap C \multimap B \otimes C \quad (i)$$

$$\geq A \otimes C \multimap B \otimes C \quad (\text{Uncurry}).$$

So (iii) holds and then (iv) follows using (Comm). (v), (vi), (vii) and (viii) follow immediately from the definition of  $\simeq$ , (i), (ii), (iii) and (iv).  $\blacksquare$

**Lemma 3.1.4** *Let the theory  $T$  extend  $\mathbf{AL}_m$ . For any formulas  $A$ ,  $B$  and  $C$ , the following hold:*

$$(A \otimes B) \otimes C \simeq_T A \otimes (B \otimes C)$$

$$A \otimes 0 \simeq_T A$$

**Proof:** For any  $D$ , using (Curry) and (Uncurry), we have:

$$(A \otimes B) \otimes C \multimap D \simeq A \multimap B \multimap C \multimap D$$

But we also have  $B \multimap C \multimap D \simeq B \otimes C \multimap D$ . Since  $\simeq$  is a congruence, using (Curry) and (Uncurry) again, we have

$$A \multimap B \multimap C \multimap D \simeq A \multimap B \otimes C \multimap D \simeq A \otimes (B \otimes C) \multimap D$$

Taking  $D$  to be  $A \otimes (B \otimes C)$  and  $(A \otimes B) \otimes C$ , we obtain  $(A \otimes B) \otimes C \geq A \otimes (B \otimes C)$  and  $A \otimes (B \otimes C) \geq (A \otimes B) \otimes C$ , i.e.,  $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ . We leave the second part as an exercise. ■

### 3.2 Semantics

We now give a semantics for the language  $\mathcal{L}$  in which the semantic values of formulas are elements of pocrim. It is convenient in describing the semantics to work in a single language including the constant 1. The value of 1 is only required to be an annihilator when that is stated explicitly.

So given a pocrim  $\mathbf{P}$ , and an assignment  $\alpha : \text{Var} \cup \{1\} \rightarrow P$  of elements of  $P$  to variables and the constant 1, we extend  $\alpha$  to a meaning function  $v_\alpha : \mathcal{L} \rightarrow P$  by interpreting  $\otimes$  and  $\multimap$  as  $+$  and  $\rightarrow$  respectively. So, for example, the formula 0, i.e.,  $1 \multimap 1$ , will be interpreted as  $\alpha(1) \rightarrow \alpha(1)$ , i.e., 0, the identity element of  $\mathbf{P}$ . We say that  $\alpha$  *satisfies* a formula  $A$ , if  $v_\alpha(A) = 0$ . We say that  $A$  is *valid* in  $\mathbf{P}$  if it is satisfied by every assignment  $\alpha : \text{Var} \cup \{1\} \rightarrow P$ . If  $\mathbf{P}$  is bounded with annihilator 1, we say  $A$  is *boundedly valid* if it is satisfied by every assignment  $\alpha : \text{Var} \cup \{1\} \rightarrow P$  such that  $\alpha(1) = 1$ . If  $\mathcal{C}$  is a class of pocrim that are not all bounded, we say a formula  $A$  is *valid* in  $\mathcal{C}$ , if it is valid in every  $\mathbf{P} \in \mathcal{C}$ . If  $\mathcal{B}$  is a class of bounded pocrim, we say a formula  $A$  is *valid* in  $\mathcal{B}$  if it is boundedly valid in every  $\mathbf{P} \in \mathcal{B}$ . (This technical trick is convenient for the statement of the theorem that follows.) A logic  $L$  is *sound* for  $\mathcal{C}$  if every  $A$  that is provable in  $L$  is valid in  $\mathcal{C}$ .  $L$  is *complete* for  $\mathcal{C}$  if every formula that is valid in  $\mathcal{C}$  is provable in  $L$ . If  $\mathbf{P}$  is a pocrim, we write  $\text{Th}(\mathbf{P})$  for the set of all formulas that are valid in  $\mathbf{P}$ . In the proof of the following theorem, we exhibit a pocrim  $\mathbf{T}$  such that  $\text{Th}(\mathbf{T})$  comprises precisely the set of formulas that are provable in  $\mathbf{AL}_m$ .

**Theorem 3.2.1** *Each of our nine logics is sound and complete for the corresponding class of pocrims shown in the following table:*

<b>AL<sub>m</sub></b>	<i>pocrims</i>
<b>AL<sub>i</sub></b>	<i>bounded pocrims</i>
<b>AL<sub>c</sub></b>	<i>involutive pocrims</i>
<b>LL<sub>m</sub></b>	<i>hoops</i>
<b>LL<sub>i</sub></b>	<i>bounded hoops</i>
<b>LL<sub>c</sub></b>	<i>involutive hoops</i>
<b>ML</b>	<i>idempotent hoops</i>
<b>IL</b>	<i>bounded idempotent hoops</i>
<b>BL</b>	<i>involutive idempotent hoops</i>

**Proof:** We give the proof for **AL<sub>m</sub>**. The modifications to give the proofs for the other logics are straightforward.

For soundness, it suffices to show that all instances of the axiom schemata used to define **AL<sub>m</sub>** are valid and that *modus ponens* preserves validity. We will just consider the axiom schema (Curry) and leave the rest as an exercise. For (Curry), we have to show that  $v_\alpha((A \otimes B \multimap C) \multimap (A \multimap B \multimap C)) = 0$  for any formulas  $A$ ,  $B$  and  $C$  and any assignment  $\alpha$  in any pocrim. Now  $v_\alpha((A \otimes B \multimap C) \multimap (A \multimap B \multimap C)) = (v_\alpha(A) + v_\alpha(B) \rightarrow v_\alpha(C)) \rightarrow (v_\alpha(A) \rightarrow v_\alpha(B) \rightarrow v_\alpha(C))$ . Hence it is sufficient to show that every pocrim satisfies  $(a + b \rightarrow c) \rightarrow (a \rightarrow b \rightarrow c) = 0$ , i.e., that every pocrim satisfies  $a + b \rightarrow c \geq a \rightarrow b \rightarrow c$ . Two applications of the residuation property (and some rearrangement using the commutative monoid laws) show that this holds iff  $(a + b) + (a + b \rightarrow c) \geq c$ , which has the form  $x + (x \rightarrow y) \geq y$ . By the residuation property, this is equivalent to  $x \rightarrow y \geq x \rightarrow y$ , which holds since  $\geq$  is a partial order, completing the proof that all instances of (Curry) are valid.

As for completeness, by Lemmas 3.1.1 and 3.1.3, we may define a structure **T** =  $(T; 0, +, \rightarrow)$  by taking  $T = \mathcal{L}/\simeq$  (the set of  $\simeq$ -equivalence classes) and defining:

$$\begin{aligned}
0 &= [0] \\
[A] + [B] &= [A \otimes B] \\
[A] \rightarrow [B] &= [A \multimap B].
\end{aligned}$$

**T** is the *term model* for **AL<sub>m</sub>**. It now follows using (Comp), (Comm), (Curry), (Uncurry) and (Wk) and Lemmas 3.1.1, 3.1.3 and 3.1.4 that **T** is a pocrim. Now as our axiom schemata are closed under substitution, a

formula  $A$  is valid in  $\mathbf{T}$  iff it is valid under the interpretation that maps each variable  $P$  to  $[P]$ , i.e., iff  $[A] = 0$ , which holds iff  $A$  is provable in  $\mathbf{AL}_m$ . Completeness follows, since if  $A$  is valid in all pocrim, then it is certainly valid in  $\mathbf{T}$ . ■

As we have defined it, the class of pocrim is a quasivariety, i.e., its defining properties are Horn clauses over equational atoms. It is known that this is the best we can do: the class of involutive pocrim cannot be characterised by equational laws. Since involutive pocrim are characterised over bounded pocrim and over pocrim by equational laws, it follows that the class of pocrim and the class of bounded pocrim are also not varieties. See [21] and the works cited therein for these results.

In [2] we present a number of proofs derived from machine-oriented derivations found by the automated theorem-prover Prover9. Our use of Prover9 relies heavily on the fact that the class of hoops is actually a variety with quite a short and simple equational axiomatisation. In a *tour de force* of equational reasoning, Bosbach [6] gave a direct proof of an equational axiomatization of the class of hoops. Using Theorem 3.2.1, we can give a new proof that hoops form a variety by showing how to transform a proof of a formula  $A$  in  $\mathbf{LL}_m$  into an equational proof that  $a = 0$ , where  $a$  is a translation into the language of pocrim of the formula  $A$ .

**Theorem 3.2.2** *Let  $\mathbf{H}$  be a structure for the signature  $(0, +, \rightarrow)$ .*

**I.**  *$\mathbf{H}$  is a hoop iff  $(H; 0, +)$  is a commutative monoid and  $\mathbf{H}$  satisfies the following equations:*

1.  $x \rightarrow x = 0$
2.  $x \rightarrow 0 = 0$
3.  $x + y \rightarrow z = x \rightarrow y \rightarrow z$
4.  $x + (x \rightarrow y) = y + (y \rightarrow x)$

**II.**  *$\mathbf{H}$  is a bounded hoop iff it satisfies the above equations and also:*

5.  $1 \rightarrow x = 0$ .

**Proof:** **I:** It follows easily from the definitions (or from Theorem 3.2.1) that equations 1 to 4 hold in any hoop. For the converse, Theorem 3.2.1 implies that it is sufficient to show that, if there is proof of  $A$  in  $\mathbf{LL}_m$  then  $[A]$ , (the element of the term model of  $\mathbf{LL}_m$  represented by  $A$ ) can be reduced to 0 using the commutative monoid laws and equations 1 to 4. It

follows that if  $a$  is the formula obtained from  $A$  by replacing  $\otimes$  and  $\multimap$  by  $+$  and  $\rightarrow$  respectively, then  $A$  is provable in  $\mathbf{LLm}$  iff every hoop satisfies  $a = 0$ . We will show how to translate a proof of  $A$  in  $\mathbf{LLm}$  into a sequence of equations  $a = a_1 = \dots = a_n = 0$ , where each equation  $a_i = a_{i+1}$  is obtained by applying one of the equations 1 to 4 to a subterm of  $a_i$  or  $a_{i+1}$  or by using the commutative monoid laws. The equational derivation is defined by recursion over a proof constructed using *modus ponens* from the axiom schemata (Comp), (Comm), (Curry), (Uncurry) and (Wk). We give the justification as we define each step in the derivation, so once the definition is complete, the proof is complete.

(Comp): we want  $(a \rightarrow b) \rightarrow (b \rightarrow c) \rightarrow (a \rightarrow c) = 0$  for arbitrary  $a, b$  and  $c$ :

$$\begin{aligned}
(a \rightarrow b) \rightarrow (b \rightarrow c) \rightarrow (a \rightarrow c) &= (a \rightarrow b) + (b \rightarrow c) + a \rightarrow c && 2 \times (\text{eq. 3}) \\
&= a + (a \rightarrow b) + (b \rightarrow c) \rightarrow c && (\text{comm. monoid}) \\
&= b + (b \rightarrow a) + (b \rightarrow c) \rightarrow c && (\text{eq. 4}) \\
&= (b \rightarrow a) + b + (b \rightarrow c) \rightarrow c && (\text{comm. monoid}) \\
&= (b \rightarrow a) + c + (c \rightarrow b) \rightarrow c && (\text{eq. 4}) \\
&= (b \rightarrow a) + (c \rightarrow b) + c \rightarrow c && (\text{comm. monoid}) \\
&= (b \rightarrow a) + (c \rightarrow b) \rightarrow c \rightarrow c && (\text{eq. 3}) \\
&= (b \rightarrow a) + (c \rightarrow b) \rightarrow 0 && (\text{eq. 1}) \\
&= 0. && (\text{eq. 2})
\end{aligned}$$

(Comm): we want  $a + b \rightarrow b + a = 0$  for arbitrary  $a$  and  $b$ :

$$\begin{aligned}
a + b \rightarrow b + a &= a + b \rightarrow a + b && (\text{comm. monoid}) \\
&= 0 && (\text{eq. 1})
\end{aligned}$$

(Curry): we want  $(a + b \rightarrow c) \rightarrow (a \rightarrow b \rightarrow c) = 0$  for arbitrary  $a, b$  and  $c$ :

$$\begin{aligned}
(a + b \rightarrow c) \rightarrow (a \rightarrow b \rightarrow c) &= (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b \rightarrow c) && (\text{eq. 3}) \\
&= 0. && (\text{eq. 1})
\end{aligned}$$

(Uncurry): we want  $(a \rightarrow b \rightarrow c) \rightarrow (a + b \rightarrow c) = 0$  for arbitrary  $a, b$  and  $c$ :

$$\begin{aligned}
(a \rightarrow b \rightarrow c) \rightarrow (a + b \rightarrow c) &= (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b \rightarrow c) && (\text{eq. 3}) \\
&= 0. && (\text{eq. 1})
\end{aligned}$$



(Wk): we want  $a + b \rightarrow a = 0$  for arbitrary  $a$  and  $b$ :

$$\begin{aligned}
a + b \rightarrow a &= b + a \rightarrow a && \text{(comm. monoid)} \\
&= b \rightarrow a \rightarrow a && \text{(eq. 3)} \\
&= b \rightarrow 0 && \text{(eq. 1)} \\
&= 0. && \text{(eq. 2)}
\end{aligned}$$

*Modus ponens*: we are given  $a = 0$  and  $a \rightarrow b = 0$  and we want  $b = 0$ :

$$\begin{aligned}
b &= b + 0 && \text{(comm. monoid)} \\
&= b + (b \rightarrow 0) && \text{(eq. 2)} \\
&= 0 + (0 \rightarrow b) && \text{(eq. 4)} \\
&= 0 \rightarrow b && \text{(comm. monoid)} \\
&= a \rightarrow b && \text{(given)} \\
&= 0. && \text{(given)}
\end{aligned}$$

This completes the recursive definition.

**II:** Like part **I**, using equation (5) to translate the axiom schema (EFQ). ■

We have chosen equations 1 to 5 for convenience in the above proof. In fact, our Prover9 work uses  $x+1 = 1$  to characterize 1 rather than  $1 \rightarrow x = 0$ . The reader may enjoy showing that the two are equivalent. A more intricate exercise is to show that equation 2 is redundant, as it can be derived from the commutative monoid laws and equations 1, 3 and 4.

In [2] we give a sequent calculus that is equivalent to our Hilbert-style presentation of **LLm**. The sequent calculus proofs can also be translated into equational proofs along similar lines to the translation given above.

## 4 Identities in Hoops

Blok and Ferreirim proved that the quasi-equational theory of hoops is decidable [5]. However, the proof does not lead to any bounds on the complexity of the decision procedure. More recently, Bova and Montagna [7] have shown that the quasi-equational theory of commutative GBL-algebras is in PSPACE and, in fact, is PSPACE-complete. It can be shown that the quasi-equational theory of commutative GBL-algebras is a conservative extension of that of hoops and hence Bova and Montagna's work implies that the quasi-equational theory of hoops is in PSPACE. Bova and Montagna use

a generalisation of the ordinal sum construction. This *poset sum* construction takes as input a family of commutative GBL-algebras  $\mathbf{G}_p$  indexed by a poset  $\mathbf{P}$ . They show that a quasi-equation involving  $n$  symbol occurrences holds in all commutative GBL-algebras iff it holds in all finite algebras of size at most  $2^{3n^2}$  that are the poset sum of a family of finite MV-chains<sup>6</sup> indexed by a poset comprising a tree of height at most  $n$  and with at most  $2^{n^2}$  nodes. They then give an ingenious non-deterministic algorithm that checks using polynomial space whether a given quasi-equation can be refuted in the corresponding set of finite algebras. Since co-NPSPACE, NPSPACE and PSPACE coincide, this shows that the quasi-equational theory is in PSPACE.

Unfortunately, the algorithm of Bova and Montagna is infeasible, certainly for hand calculation even on small examples: the valid equation  $\neg(\neg\neg x \rightarrow x) = 0$  contains 8 symbols and just the number of trees to be considered would be enormous. As we are interested in verifying certain specific identities, we need a more practical method. To this end, we will show that an identity is valid in all hoops iff it is valid in a restricted class of hoops enjoying some very convenient algebraic properties. This does not provide a decision procedure, but it does provide a quick indirect method of proof for many important identities. We begin with a review of the decision problem for identities in involutive hoops, which we often need to consider when applying the indirect method for bounded hoops in general.

#### 4.1 Identities in involutive hoops

By contrast with the case for general hoops, the equational theory of involutive hoops can be decided by a computationally efficient reduction to (linear) real arithmetic. In this section, we review the proof of this result, which relies on the fact that involutive hoops are definitionally equivalent to MV-algebras. This definitional equivalence is stated without proof in [5]. We give the proof here for expository purposes and because it involves some identities that will be useful later.

MV-algebras were originally introduced by Chang [10] and have been widely studied. We adopt the definition and notation of [12]:

**Definition 4.1.1** *An MV-algebra is a structure for the signature  $(0, \oplus, \neg)$  whose  $(0, \oplus)$ -reduct is a commutative monoid and which, with  $x \ominus y$  defined*

---

<sup>6</sup> Finite MV-chains are the MV-algebras corresponding to the hoops  $\mathbf{L}_i$  of Example 2.2.3.

as  $\neg(\neg x \oplus y)$ , satisfies the following identities:

$$\begin{aligned} x \oplus \neg 0 &= \neg 0 \\ \neg \neg x &= x \\ (y \ominus x) \oplus x &= (x \ominus y) \oplus y \end{aligned}$$

**Lemma 4.1.1** *If  $\mathbf{H}$  is an involutive hoop, then  $\mathbf{H}$  satisfies:*

$$x \rightarrow y = \neg(x + \neg y) \quad x + y = \neg(x \rightarrow \neg y).$$

**Proof:** Recalling that  $\neg x = x \rightarrow 1$  by definition and using [dne] we have:

$$x \rightarrow y = x \rightarrow \neg \neg y = x \rightarrow \neg y \rightarrow 1 = x + \neg y \rightarrow 1 = \neg(x + \neg y),$$

and then we have  $\neg(x \rightarrow \neg y) = \neg \neg(x + \neg y) = x + y$ .  $\blacksquare$

**Theorem 4.1.2** *The variety of involutive hoops and the variety of MV-algebras are definitionally equivalent.*

**Proof:** Let  $\mathbf{H}$  be an involutive hoop and define  $x \oplus y = x + y$  and  $\neg x = x \rightarrow 1$ . Then  $(0, \oplus)$  is a commutative monoid and we have  $x \oplus \neg 0 = x + 1 = 1 = \neg 0$  and  $\neg \neg x = x$ . Moreover, by Lemma 4.1.1,  $x \ominus y = \neg(\neg x \oplus y) = y \rightarrow x$  and hence  $(y \ominus x) \oplus x = x + (x \rightarrow y) = y + (y \rightarrow x) = (x \ominus y) \oplus y$ . Thus  $(H; 0, \oplus, \neg)$  is an MV-algebra. Conversely, let  $\mathbf{M}$  be an MV-algebra and define  $1 = \neg 0$ ,  $x + y = x \oplus y$  and  $x \rightarrow y = \neg(x + \neg y)$ . Then certainly  $(M; 0, +)$  is a commutative monoid. We have  $x \rightarrow 1 = \neg(x + \neg \neg 0) = \neg x$ . Hence  $x + \neg x = x + (x \rightarrow 1) = 1 + (1 \rightarrow x) = 1$ , so that  $x \rightarrow x = \neg(x + \neg x) = \neg 1 = 0$ . Hence equation 1 in the equational characterization of bounded hoops given in Theorem 3.2.2 is satisfied. The other equations in that characterization are easily verified and so, as we have  $\neg \neg x = x$ ,  $(M; 0, +, \rightarrow)$  is an involutive hoop.  $\blacksquare$

Chang [10, 11] showed that the system  $\mathbf{L}$  of Section 2.1 is sound and complete for the class of MV-algebras under a semantics which corresponds to our semantics for hoops under the equivalence of Theorem 4.1.2. Since  $\mathbf{LL}_{\mathbf{C}}$  is sound and complete for involutive hoops, it follows that  $\mathbf{LL}_{\mathbf{C}}$  and  $\mathbf{L}$  are equivalent. Chang's work also implies that an identity holds in all MV-algebras iff it holds in the MV-algebra corresponding to the involutive hoop  $[0, 1]$ . (See [12] for more information on MV-algebras.)

**Theorem 4.1.3** *An identity  $s = t$  holds in all involutive hoops iff it holds in the hoop  $[0, 1]$  of Example 2.2.2. Hence the equational theory of involutive hoops is decidable.*

**Proof:** The first claim follows from the remarks above about the equivalence between involutive hoops and MV-algebras and the definition of  $[0, 1]$ . Given the first claim, to decide  $s = t$ , use the formulas for the operations on  $[0, 1]$  given in Example 2.2.2 to translate  $s = t$  into a formula in the language of real arithmetic, treating  $\max$  and  $\min$  as abbreviations:  $\phi(\max(x, y)) \equiv (x \geq y \wedge \phi(x)) \vee (x < y \wedge \phi(y))$  and  $\phi(\min(x, y)) \equiv (x \geq y \wedge \phi(y)) \vee (x < y \wedge \phi(x))$ . Equality of the translated formula may then be decided by the well-known decision procedures for (linear) real arithmetic. ■

It can be shown, using results of Blok and Ferreirim [5], that an identity holds in all Wajsberg hoops iff it holds in the hoop  $\mathbb{R}^{\geq 0}$  of Example 2.2.4. Hence the equational theory of Wajsberg hoops also reduces to (linear) real arithmetic.

## 4.2 Identities in general hoops

We now give our indirect method for proving identities in general hoops. The method is based on the characterization of subdirectly irreducible hoops due to Blok and Ferreirim [5]. They proved that a hoop  $\mathbf{H}$  is subdirectly irreducible iff it is isomorphic to an ordinal sum  $\mathbf{S} \frown \mathbf{F}$  where  $\mathbf{S}$  is subdirectly irreducible, totally ordered and Wajsberg and where  $\mathbf{S}$  is trivial iff  $\mathbf{H}$  is trivial.  $\mathbf{S}$  and  $\mathbf{F}$  are uniquely determined by these conditions and are called the *support* and the *fixed subhoop* of  $\mathbf{H}$  respectively.

The following theorem is really two: one for bounded hoops and one for all hoops. From now on, when we work with bounded hoops, we will take the annihilator 1 as part of the signature, so that homomorphisms must preserve it and it must be included when we consider the bounded subhoop of a given bounded hoop  $\mathbf{H}$  generated by some subset of  $H$ .

**Theorem 4.2.1** *Let  $\phi$  be an identity in the language of a (bounded) hoop in the variables  $x_1, \dots, x_n$ . Then  $\forall x_1, \dots, x_n \cdot \phi$  is valid in the class of all (bounded) hoops iff  $\phi(x_1, \dots, x_n)$  holds under any interpretation of  $x_1, \dots, x_n$  in a (bounded) hoop  $\mathbf{H}$  that can be expressed as an ordinal sum  $\mathbf{S} \frown \mathbf{F}$  where  $\mathbf{S}$  is subdirectly irreducible and Wajsberg, where  $\mathbf{H}$  is generated by  $x_1, \dots, x_n$  and where  $S = \{0\}$  iff  $H = \{0\}$ .*

**Proof:**  $\Rightarrow$ : if  $\forall x_1, \dots, x_n \cdot \phi$  is valid in the class of all (bounded) hoops, then  $\phi(x_1, \dots, x_n)$  holds for any interpretation of  $x_1, \dots, x_n$  in any (bounded) hoop.

$\Leftarrow$ : Assume that  $\phi(x_1, \dots, x_n)$  holds in any (bounded) hoop  $\mathbf{H}$  satisfying the stated conditions on  $\mathbf{H}$  and on the interpretation of  $x_1, \dots, x_n$ . Let then

To verify  $\phi(x_1, \dots, x_n)$  in all hoops, verify it in the following cases:  
*Case (i):  $\mathbf{H} \cong \mathbf{S} \frown \mathbf{F}$  with  $F = \{0\}$ .  $\mathbf{H} \cong \mathbf{S}$ .*  
*Case (ii):  $\mathbf{H} \cong \mathbf{S} \frown \mathbf{F}$  with  $F \neq \{0\}$ . There is a subcase for each choice of  $I = \{i \mid x_i \in S\} \neq \emptyset$  and  $J = \{j \mid x_j \in F\} \neq \emptyset$ , with  $\mathbf{F}$  generated by the  $x_j$  with  $j \in J$ .*  
*In both cases  $\mathbf{S}$  is subdirectly irreducible, Wajsberg and generated by the  $x_i \in S$ .*

Figure 4: Template for applying theorem 4.2.1 to all hoops

$\mathbf{H}$  be an arbitrary (bounded) hoop and let  $x_1, \dots, x_n \in H$ . We must prove that  $\phi(x_1, \dots, x_n)$  holds in  $\mathbf{H}$ . Clearly  $\phi(x_1, \dots, x_n)$  holds in  $\mathbf{H}$  iff it holds in the subhoop of  $\mathbf{H}$  generated by  $x_1, \dots, x_n$ . So we may assume  $\mathbf{H}$  is generated by  $x_1, \dots, x_n$ . By a classic result of Birkhoff (e.g., see [9, Theorem II.8.6])  $\mathbf{H}$  is isomorphic to a subdirect product of subdirectly irreducible hoops each of which is a homomorphic image of  $\mathbf{H}$  (and hence is generated by the images  $[x_1], \dots, [x_n]$  of  $x_1, \dots, x_n$ ). The identity  $\phi(x_1, \dots, x_n)$  holds in the subdirect product if  $\phi([x_1], \dots, [x_n])$  holds in each factor. So we may assume that  $\mathbf{H}$  is subdirectly irreducible and generated by  $x_1, \dots, x_n$ . By the theorem of Blok and Ferreirim,  $\mathbf{H}$  has subhoops  $\mathbf{S}$  and  $\mathbf{F}$  such that  $\mathbf{S}$  is subdirectly irreducible and Wajsberg and  $\mathbf{H} \cong \mathbf{S} \frown \mathbf{F}$ . Moreover  $S = \{0\}$  iff  $H = \{0\}$ . Hence, by assumption,  $\phi(x_1, \dots, x_n)$  holds in  $\mathbf{H}$ . ■

By the definition of the ordinal sum, if  $s \in S \setminus \{0\}$  and  $f \in F \setminus \{0\}$ , then  $(f \rightarrow s) \rightarrow s = 0 \rightarrow s = s \neq (s \rightarrow f) \rightarrow f = f \rightarrow f = 0$ , i.e.,  $s$  and  $f$  do not satisfy the Wajsberg condition. So, if  $\mathbf{H}$  is subdirectly irreducible and Wajsberg, then it is equal to its support  $\mathbf{S}$  and hence is totally ordered.

The rest of Section 4 illustrates the use of Theorem 4.2.1. To prove that an identity  $\phi(x_1, \dots, x_n)$  holds in all hoops, we follow the template of Figure 4. We consider an interpretation of the variables  $x_1, \dots, x_n$  in a hoop  $\mathbf{H} = \mathbf{S} \frown \mathbf{F}$  satisfying the stated conditions. Apart from the trivial case when  $n = 0$ , the set  $I = \{i \mid x_i \in S\}$  cannot be empty (otherwise we would have  $S = \{0\}$  while  $H = F \neq \{0\}$ ). We then consider all possible cases for the set  $I$ . So let  $J = \{1, \dots, n\} \setminus I$ . If  $J = \emptyset$ , then we must verify that  $\phi$  holds in a subdirectly irreducible Wajsberg hoop  $\mathbf{S}$  (Case (i)). If  $J \neq \emptyset$ , we have to verify that  $\phi$  holds in  $\mathbf{S} \frown \mathbf{F}$ , where  $\mathbf{S}$  is generated by the  $x_i$  with  $i \in I$ ,  $\mathbf{F}$  is generated by the  $x_j$  with  $j \in J$  and  $x_j \neq 0$  for  $j \in J$  (Case (ii)). The cases where  $J \neq \emptyset$  are often easy to verify using identities such as  $x_j \rightarrow x_i = 0$  and  $x_i \rightarrow x_j = x_j$  when  $i \in I$  and  $j \in J$  that follow from

the definition of the ordinal sum.

To prove an identity holds in all bounded hoops, we follow the template of Figure 5. We have the same cases as for the unbounded case with the extra assumption that  $\mathbf{H}$  is bounded, and hence involutive in Case (i). We must also consider the possibility that  $J = \emptyset$  and  $F$  is generated by the constant 1, i.e.  $\mathbf{H} \cong \mathbf{S} \frown \mathbb{B}$  where  $\mathbf{S}$  is subdirectly irreducible and Wajsberg and  $\mathbb{B}$  is the boolean hoop with  $B = \{0, 1\}$  (Case (iii)).

To verify  $\phi(x_1, \dots, x_n)$  in all bounded hoops, verify it in the following cases:  
 Cases (i) and (ii): As in Figure 4, with the extra assumption that  $\mathbf{H}$  is bounded.  
 Case (iii):  $\mathbf{H} \cong \mathbf{S} \frown \mathbb{B}$ , with all  $x_i \in S$ .  
 In all cases,  $\mathbf{S}$  is subdirectly irreducible, Wajsberg and generated by the  $x_i \in S$ .

Figure 5: Template for applying theorem 4.2.1 to bounded hoops

The structure of free MV-algebras and hence of free involutive hoops is quite well understood. See [12] for a good account of this topic. Very little is known about free hoops or free bounded hoops. It can be shown that the free bounded hoop on one generator is a subdirect product of hoops isomorphic to subhoops of  $[0, 1]$ ,  $[0, 1] \frown \mathbb{B}$  and  $\mathbb{R}^{\geq 0} \frown \mathbb{B}$ . This suggested the identity of the following example.

**Example 4.2.1** *If  $0 < k \in \mathbb{N}$ , the identity  $\neg kx \rightarrow \neg \neg x \rightarrow x = 0$  clearly holds in any involutive hoop. It also holds in any hoop of the form  $\mathbf{S} \frown \mathbb{B}$  (since in such a hoop, either  $x = 1$  or  $\neg kx = 1$ ). This covers cases (i) and (iii) in the template of Figure 5. As the identity has only one variable, there is nothing to prove in case (ii). Hence,  $\neg kx \rightarrow \neg \neg x \rightarrow x = 0$  holds in any bounded hoop. In [2], we demonstrate how to construct a proof in  $\mathbf{LL}_1$  of the formula of  $\mathcal{L}$  that corresponds to this identity. Unfolding the constructions used in the inductive step of this demonstration reveals 19 intricate applications of the axiom (CWC).*

### 4.3 Application: de Morgan identities

In any bounded pocrim, we have  $\neg(x+y) = x+y \rightarrow 1 = x \rightarrow y \rightarrow 1 = x \rightarrow \neg y$ , so the first identity in the following theorem is easily proved. In a bounded hoop, we have a kind of dual identity:  $\neg(x \rightarrow y) = \neg \neg x + \neg y$ . As can be

seen in [2], the simplest known elementary proof of the dual identity is quite involved. The indirect proof using Theorem 4.2.1 is much shorter:

**Theorem 4.3.1** *The following identities are satisfied in any bounded hoop:*

$$\neg(x + y) = x \rightarrow \neg y \quad \neg(x \rightarrow y) = \neg\neg x + \neg y$$

**Proof:** See the above remarks for the first identity. For the second we follow the template of Figure 5, requiring us to prove the identity in the following cases for a hoop  $\mathbf{H}$  and its elements  $x$  and  $y$ :

Case (i): Our assumptions imply that  $\mathbf{H}$  is involutive, hence by Lemma 4.1.1

$$\neg(x \rightarrow y) = \neg\neg(x + \neg y) = x + \neg y = \neg\neg x + \neg y$$

Case (ii):  $\mathbf{H} = \mathbf{S} \cap \mathbf{F}$ ,  $\{x, y\} \cap S \neq \emptyset$ ,  $\{x, y\} \cap F \setminus \{0\} \neq \emptyset$ : this leads to two subcases that are proved using elementary properties of  $\mathbf{S} \cap \mathbf{F}$ , as follows:

Subcase (ii)(a):  $x \in S$ ,  $y \in F \setminus \{0\}$ :

$$\neg(x \rightarrow y) = \neg y = 0 + \neg y = \neg\neg x + \neg y$$

Subcase (ii)(b):  $x \in F \setminus \{0\}$ ,  $y \in S$ :

$$\neg(x \rightarrow y) = \neg 0 = 1 = \neg\neg x + 1 = \neg\neg x + \neg y.$$

Case (iii):  $\mathbf{H} = \mathbf{S} \cap \mathbb{B}$  where  $x, y \in S$ : for  $u \in S$ ,  $\neg u = 1$ , so as  $x \rightarrow y \in S$ , we have  $\neg(x \rightarrow y) = 1 = 0 + 1 = \neg\neg x + \neg y$ .  $\blacksquare$

#### 4.4 Application: The Ferreirim-Veroff-Spinks theorem

Ferreirim [13] proved by indirect methods that if  $e$  is an idempotent element in a  $k$ -potent hoop, then the mapping  $x \mapsto e \rightarrow x$  is an additive homomorphism. Using Otter [19], Veroff and Spinks [24] found a syntactic proof without assuming  $k$ -potency. A simplified and more abstract presentation of their proof is given by the present authors in [2]. When we express the theorem as an identity and apply Theorem 4.2.1, it turns out that it is only the case when the hoop is subdirectly irreducible and Wajsberg that presents any difficulties:

**Theorem 4.4.1** *The following identity holds in any hoop:*

$$(e \rightarrow e + e) \rightarrow (e \rightarrow x + y) \rightarrow (e \rightarrow x) + (e \rightarrow y) = 0.$$

**Proof:** Writing  $a = e \rightarrow e + e$ ,  $b = e \rightarrow x + y$  and  $c = (e \rightarrow x) + (e \rightarrow y)$ , what we have to prove is that  $a \rightarrow b \rightarrow c = 0$ . According to the template of Figure 4, it is sufficient to verify  $a \rightarrow b \rightarrow c = 0$  in the following cases for a hoop  $\mathbf{H}$  and its elements  $e$ ,  $x$  and  $y$ :

Case (i):  $\mathbf{H}$  subdirectly irreducible and Wajsberg: In this case, we claim that  $a + b \geq c$ , whence  $a \rightarrow b \rightarrow c = 0$  as required. By Lemma 4.4.2, there are two subcases:

Subcase (i)(a):  $e \rightarrow e + e = e$ : we have  $a = e$ , so that:

$$a + b = e + (e \rightarrow x + y) \geq x + y \geq (e \rightarrow x) + (e \rightarrow y) = c.$$

Subcase (i)(b):  $\mathbf{H}$  is bounded and  $e + e = 1$ : we have  $a = e \rightarrow 1$ , so that:

$$a + b = (e \rightarrow 1) + (e \rightarrow x + y) \geq (e \rightarrow x) + (e \rightarrow y) = c.$$

Case (ii):  $\mathbf{H} = \mathbf{S} \cap \mathbf{F}$ ,  $\{e, x, y\} \cap S \neq \emptyset$ ,  $\{e, x, y\} \cap F \setminus \{0\} \neq \emptyset$ : since the equation is symmetric in  $x$  and  $y$ , this leads to 4 subcases. In each of these subcases, we claim that  $b = c$ , whence  $a \rightarrow b \rightarrow c = 0$  as required. The claim is verified using elementary properties of  $\mathbf{S} \cap \mathbf{F}$  as follows:

Subcase (ii)(a):  $e \in S, x, y \in F \setminus \{0\}$ :

$$b = e \rightarrow x + y = x + y = (e \rightarrow x) + (e \rightarrow y) = c.$$

Subcase (ii)(b):  $e, x \in S, y \in F \setminus \{0\}$ :

$$b = e \rightarrow x + y = e \rightarrow y = y = (e \rightarrow x) + y = (e \rightarrow x) + (e \rightarrow y) = c.$$

Subcase (ii)(c):  $e, x \in F \setminus \{0\}, y \in S$ :

$$b = e \rightarrow x + y = e \rightarrow x = (e \rightarrow x) + 0 = (e \rightarrow x) + (e \rightarrow y) = c.$$

Subcase (ii)(d):  $e \in F \setminus \{0\}, x, y \in S$ :

$$b = e \rightarrow x + y = 0 = 0 + 0 = (e \rightarrow x) + (e \rightarrow y) = c. \quad \blacksquare$$

**Lemma 4.4.2** *If  $e$  is any element of a totally ordered Wajsberg hoop  $\mathbf{H}$  then either  $e \rightarrow e + e = e$  or  $\mathbf{H}$  is bounded and  $e + e = 1$ .*

**Proof:** We have  $e + e = e + e + (e + e \rightarrow e) = e + (e \rightarrow e + e)$ , so the lemma follows from the fact that totally ordered irreducible Wajsberg hoops are *semi-cancellative*, i.e., if  $a + b = a + c$  with  $b \neq c$ , then  $\mathbf{H}$  is bounded and  $a + b = 1$ . This can be extracted from the characterization of subdirectly irreducible hoops: see [1, Theorem 26, part (ix)]. (The theorem we state



there is for a special class of hoops called coops, but the proof of that part of the theorem goes through in the same way for a general hoop.) However, Ferreirim [13, Lemma 4.5] gives the following neat elementary proof: assume  $a + b = a + c$  is not an annihilator, so there is  $u \in H$  with  $u > a + b$ . Since  $u > a + b$ ,  $b > a \rightarrow u$  is impossible, so, as  $\mathbf{H}$  is totally ordered, we must have  $a \rightarrow u \geq b$ , i.e.,  $(a \rightarrow u) \rightarrow b = 0$ . But then, as  $\mathbf{H}$  is Wajsberg, we have:

$$\begin{aligned} b &= ((a \rightarrow u) \rightarrow b) \rightarrow b \\ &= (b \rightarrow a \rightarrow u) \rightarrow a \rightarrow u \\ &= a \rightarrow (a + b \rightarrow u) \rightarrow u \\ &= a \rightarrow (u \rightarrow a + b) \rightarrow a + b \\ &= a \rightarrow a + b \end{aligned}$$

As  $a + b = a + c$ , the same argument gives us that  $c = a \rightarrow a + c$ , so  $b = c$ . ■

#### 4.5 Application: the idempotent subhoop

The set of idempotent elements of a pocrim is clearly closed under  $+$  but it need not be closed under  $\rightarrow$ . For example, in the pocrim  $\mathbf{Q}_4$  of Example 2.2.6,  $u$  and  $1$  are idempotent, but  $u \rightarrow 1 = v \neq 1$  and  $v + v = 1$ . In a hoop, however, Theorem 4.2.1 enables us to prove the following somewhat surprising theorem:

**Theorem 4.5.1** *Let  $\mathbf{H}$  be a hoop. The set  $J = \{x : H \mid x + x = x\}$  of idempotent elements of  $\mathbf{H}$  is the universe of a subhoop.*

**Proof:** We must show that  $0 \in J$ ,  $J + J \subseteq J$  and  $J \rightarrow J \subseteq J$ . The first two assertions are easy. As for  $J \rightarrow J \subseteq J$ , let us define  $i : H \rightarrow H$  by  $i(x) = x \rightarrow x + x$ , so that  $x \in J$  iff  $i(x) = 0$ . It is sufficient to show that the following identity holds:

$$i(x) \rightarrow i(y) \rightarrow i(x \rightarrow y) = 0 \quad (*)$$

According to the template of Figure 4, it is sufficient to verify  $(*)$  in the following cases for a hoop  $\mathbf{H}$  and its elements  $x$  and  $y$ :

Case (i):  $\mathbf{H}$  subdirectly irreducible and Wajsberg: By Lemma 4.4.2, if  $\mathbf{H}$  is not bounded, then  $i(x) = x$  for all  $x$  and  $(*)$  is trivial. If  $\mathbf{H}$  is bounded, then it is involutive and by Theorem 4.1.3, it is enough to verify  $(*)$  in  $[0, 1]$ . Now in  $[0, 1]$ , we have:

$$i(x) = \begin{cases} x & \text{if } x \leq \frac{1}{2} \\ 1 - x & \text{if } x \geq \frac{1}{2}. \end{cases}$$

(\*) holds in any hoop if  $x \geq y$ , so, in  $[0, 1]$ , we may assume  $x < y$ , so that  $x \rightarrow y = y - x$ . Then (\*) holds iff  $i(x) + i(y) \geq i(y - x)$  and we have eight cases for  $x, y, y - x \in [0, 1]$  as follows:

$x > \frac{1}{2}$	$y > \frac{1}{2}$	$y - x > \frac{1}{2}$	$i(x) + i(y) \geq i(y - x)$
✓	✓	✓	✘
✓	✓	×	$1 - x + 1 - y \geq y - x$
✓	×	✓	✘
✓	×	×	✘
×	✓	✓	$x + 1 - y \geq 1 - (y - x)$
×	✓	×	$x + 1 - y \geq y - x$
×	×	✓	✘
×	×	×	$x + y \geq y - x$

The cases marked ✘ are impossible, as the constraints on  $x$ ,  $y$  and  $y - x$  are inconsistent. In the other cases, the inequalities are easily verified using the constraints. That completes case (i).

Case (ii):  $\mathbf{H} = \mathbf{S} \cap \mathbf{F}$ ,  $\{x, y\} \cap S \neq \emptyset$ ,  $\{x, y\} \cap F \setminus \{0\} \neq \emptyset$ : This leads to 2 subcases. These are verified using elementary properties of  $\mathbf{S} \cap \mathbf{F}$  as follows:  
Subcase (ii)(a):  $x \in S, y \in F \setminus \{0\}$ : we have:

$$i(x) \rightarrow i(y) \rightarrow i(x \rightarrow y) = i(x) \rightarrow i(y) \rightarrow i(y) = i(x) \rightarrow 0 = 0.$$

Subcase (ii)(b):  $x \in F \setminus \{0\}, y \in S$ : we have:

$$i(x) \rightarrow i(y) \rightarrow i(x \rightarrow y) = i(x) \rightarrow i(y) \rightarrow 0 = 0.$$

This completes case (ii). ■

## 5 Double Negation Translations

In this section we undertake an algebraic study of the syntactic translations known as double negation translations (or negative translations). Throughout this section all our pocrimms will be bounded. We will view 1 as a constant in the signature for bounded pocrimms and so a homomorphism  $f$  must satisfy  $f(1) = 1$ . In the semantics, 1 will always be interpreted as 1, so assignments will be functions with domain  $\mathbf{Var}$  rather than  $\mathbf{Var} \cup \{1\}$ .

### 5.1 The Double Negation Mapping

If  $\mathbf{P}$  is a pocrim, let  $N = \text{im}(\neg) = \{\neg x \mid x \in P\}$ . Since  $\delta(\neg x) = \neg x$ ,  $\text{im}(\delta) = N$ . Clearly  $\{0, 1\} \subseteq N$  and  $N$  is closed under  $\rightarrow$ , since  $\neg x \rightarrow \neg y =$

$\neg(\neg x + y)$ . In general,  $N$  is not closed under addition and hence is not a subpocrim and  $\delta$  does not respect either  $+$  or  $\rightarrow$ :

**Example 5.1.1** *There is a pocrim  $\mathbf{U}$  with elements  $0 < a < b < c < 1$  and with  $+$ ,  $\rightarrow$  and  $\delta$  as follows:*

$+$	0	$a$	$b$	$c$	1	$\rightarrow$	0	$a$	$b$	$c$	1	$\delta$
0	0	$a$	$b$	$c$	1	0	0	$a$	$b$	$c$	1	0
$a$	$a$	$b$	$b$	1	1	$a$	0	0	$a$	$c$	$c$	$a$
$b$	$b$	$b$	$b$	1	1	$b$	0	0	0	$c$	$c$	$b$
$c$	$c$	1	1	1	1	$c$	0	0	0	0	$a$	$c$
1	1	1	1	1	1	1	0	0	0	0	0	1

So, in  $\mathbf{U}$ ,  $\delta(a \rightarrow b) = a \neq 0 = \delta(a) \rightarrow \delta(b)$ ,  $\delta(a + a) = a \neq b = \delta(a) + \delta(a)$  and  $\delta(\delta(a) + \delta(a)) \neq \delta(a) + \delta(a)$ . The image of negation is  $\{0, a, c, 1\}$ , which is not closed under addition, since  $a + a = b$ .

The situation in a hoop is much more satisfactory. To describe it, we first make the following definition:

**Definition 5.1.1** *If  $\mathbf{H}$  is a bounded hoop, the involutive replica,  $\text{IR}(\mathbf{H})$ , of  $\mathbf{H}$  is  $\mathbf{H}/\theta$ , where  $\theta$  is the smallest congruence such that  $x \theta \delta(x)$  for all  $x \in H$ .*

$\mathbf{H} \mapsto \text{IR}(\mathbf{H})$  is the objects part of a functor from the category of bounded hoops to the category of involutive hoops and every homomorphism from  $\mathbf{H}$  to an involutive hoop factors uniquely through  $\text{IR}(\mathbf{H})$ .

**Theorem 5.1.1** *If  $\mathbf{H}$  is a bounded hoop, then the double negation mapping,  $\delta$ , is a homomorphism  $\mathbf{H} \rightarrow \mathbf{H}$ . Moreover, if  $p : \mathbf{H} \rightarrow \text{IR}(\mathbf{H})$  is the natural projection, then  $p$  factors as  $p = i \circ \delta$  where  $i : \text{im}(\delta) \rightarrow \text{IR}(\mathbf{H})$  is an isomorphism.*

**Proof:** By Theorem 4.3.1, we have:

$$\begin{aligned}\delta(x) + \delta(y) &= \neg(x \rightarrow \neg y) = \delta(x + y) \\ \delta(x) \rightarrow \delta(y) &= \neg(\delta(x) + \neg y) = \delta(x) \rightarrow \delta(y)\end{aligned}$$

As  $\delta$  fixes the constants 0 and 1, this proves that  $\delta$  is a homomorphism. The claim about  $p$  is equivalent to the claim that  $\ker(\delta) = \ker(p)$ . Now  $\text{IR}(\mathbf{H})$  is the quotient  $\mathbf{H}/\theta$  where  $\theta$  is the smallest congruence such that  $x \theta \delta(x)$  for all  $x \in H$ . As  $\delta$  is an idempotent endomorphism, it is not difficult to see that  $x \theta y$  iff  $\delta(x) = \delta(y)$ . So we have  $\ker(\delta) = \{x \mid \delta(x) = 0\} = \{x \mid \delta(x) = \delta(0)\} = \{x \mid x \theta 0\} = \ker(p)$ . ■

## 5.2 Semantics for Double Negation Translations

Beginning with Kolmogorov [18], logicians have studied *double negation translations* that represent classical logic in intuitionistic logic. Kolmogorov's translation inductively replaces every subformula of a formula by its double negation. Subsequent authors have devised more economical translations: Gentzen's translation [14] applies double negation to atomic formulas only and Glivenko's translation [15] just applies double negation to a formula without changing its internal structure.

We wish to undertake an algebraic analysis of translations such as the various double negation translations. We will view the translations as variant semantics and so we need a framework to compare different semantics.

**Definition 5.2.1** *Let  $\mathbf{Poc}_1$  be the category of bounded pocrimms and homomorphisms and let  $\mathbf{Set}$  be the category of sets. Given any set  $X$ , let  $H_X : \mathbf{Poc}_1 \rightarrow \mathbf{Set}$  be the functor that maps a pocrim  $\mathbf{P}$  to  $\mathbf{Hom}_{\mathbf{Set}}(X, P)$ , i.e., the set of all functions from  $X$  to  $P$ , and maps a homomorphism  $h : \mathbf{P} \rightarrow \mathbf{Q}$  to  $f \mapsto h \circ f : \mathbf{Hom}_{\mathbf{Set}}(X, P) \rightarrow \mathbf{Hom}_{\mathbf{Set}}(X, Q)$ . Now let  $\mathbf{Ass} = H_{\mathbf{Var}}$  and  $\mathbf{Sem} = H_{\mathcal{L}}$ . We define a semantics to be a natural transformation  $\mu : \mathbf{Ass} \rightarrow \mathbf{Sem}$ .*

So given a bounded pocrim  $\mathbf{P}$ ,  $\mathbf{Ass}(\mathbf{P})$  denotes the set of assignments  $\alpha : \mathbf{Var} \rightarrow P$ , while  $\mathbf{Sem}(\mathbf{P})$  denotes the set of all possible functions  $s : \mathcal{L} \rightarrow P$ . A semantics  $\mu$  is a family of functions  $\mu_{\mathbf{P}}$  indexed by bounded pocrimms  $\mathbf{P}$  such that  $\mu_{\mathbf{P}} : \mathbf{Ass}(\mathbf{P}) \rightarrow \mathbf{Sem}(\mathbf{P})$  and such that for any homomorphism  $f : \mathbf{P} \rightarrow \mathbf{Q}$  the following diagram commutes.

$$\begin{array}{ccc} \mathbf{Ass}(\mathbf{P}) & \xrightarrow{\mathbf{Ass}(f)} & \mathbf{Ass}(\mathbf{Q}) \\ \downarrow \mu_{\mathbf{P}} & & \downarrow \mu_{\mathbf{Q}} \\ \mathbf{Sem}(\mathbf{P}) & \xrightarrow{\mathbf{Sem}(f)} & \mathbf{Sem}(\mathbf{Q}) \end{array}$$

The standard semantics  $\mu^S$  is the one used to define bounded validity in Section 3: it simply uses the given assignment  $\alpha : \mathbf{Var} \rightarrow P$  to give values to the variables in a formula in  $\mathcal{L}$  and then calculates its value interpreting  $1$ ,  $\otimes$  and  $\multimap$  as  $1$ ,  $+$  and  $\rightarrow$  respectively:

$$\begin{aligned} \mu_{\mathbf{P}}^S(\alpha)(V_i) &= \alpha(V_i) \\ \mu_{\mathbf{P}}^S(\alpha)(1) &= 1 \\ \mu_{\mathbf{P}}^S(\alpha)(A \otimes B) &= \mu_{\mathbf{P}}^S(\alpha)(A) + \mu_{\mathbf{P}}^S(\alpha)(B) \\ \mu_{\mathbf{P}}^S(\alpha)(A \multimap B) &= \mu_{\mathbf{P}}^S(\alpha)(A) \rightarrow \mu_{\mathbf{P}}^S(\alpha)(B) \end{aligned}$$

The Kolmogorov translation corresponds to a semantics  $\mu^K$  defined like  $\mu^S$ , but applying double negation to everything in sight:

$$\begin{aligned}\mu_{\mathbf{P}}^K(\alpha)(V_i) &= \delta(\alpha(V_i)) \\ \mu_{\mathbf{P}}^K(\alpha)(1) &= 1 \\ \mu_{\mathbf{P}}^K(\alpha)(A \otimes B) &= \delta(\mu_{\mathbf{P}}^K(\alpha)(A) + \mu_{\mathbf{P}}^K(\alpha)(B)) \\ \mu_{\mathbf{P}}^K(\alpha)(A \multimap B) &= \delta(\mu_{\mathbf{P}}^K(\alpha)(A) \rightarrow \mu_{\mathbf{P}}^K(\alpha)(B))\end{aligned}$$

It is easily verified that  $\mu^S$  and  $\mu^K$  are indeed natural transformations  $\text{Ass} \rightarrow \text{Sem}$ . The Gentzen and Glivenko translations correspond to semantics obtained by composing the standard semantics with double negation:

$$\begin{aligned}\mu^{\text{Gen}} &= \mu^S \circ \delta^{\text{Var}} \\ \mu^{\text{Gli}} &= \delta^{\mathcal{L}} \circ \mu^S\end{aligned}$$

where  $\delta^X$  denotes the natural transformation from  $H_X = \text{Hom}_{\text{Set}}(X, \cdot)$  to itself with  $\delta_{\mathbf{P}}^X = f \mapsto \delta \circ f$ . It is clear from Theorem 5.1.1 that the Kolmogorov, Gentzen and Glivenko semantics all agree when restricted to hoops.

**Definition 5.2.2** *Let  $\mathcal{C}$  be a class of bounded pocrimms, we say that a semantics  $\mu$  is a double negation semantics for  $\mathcal{C}$  if the following conditions hold:*

**(DNS1)** *If  $\mathbf{P} \in \mathcal{C}$  is involutive, then  $\mu_{\mathbf{P}} = \mu_{\mathbf{P}}^S$ .*

**(DNS2)** *Given a formula  $A$ , if, for every involutive  $\mathbf{P} \in \mathcal{C}$  and every  $\alpha : \text{Var} \rightarrow P$ , we have:*

$$\mu_{\mathbf{P}}^S(\alpha)(A) = 0,$$

*then, for every  $\mathbf{P} \in \mathcal{C}$  and every  $\alpha : \text{Var} \rightarrow P$ , we have:*

$$\mu_{\mathbf{P}}(\alpha)(A) = 0.$$

**(DNS3)**  $\delta^{\mathcal{L}} \circ \mu = \mu$ .

**Remark 5.2.1** *Let us write  $\text{Th}(\mathcal{C})$  for the theory of a class of pocrimms, i.e., the set of all formulas  $A$  such that  $\mu_{\mathbf{P}}^S(\alpha)(A) = 0$  for every  $\alpha : \text{Var} \rightarrow P$  where  $\mathbf{P} \in \mathcal{C}$ . The above definition can be seen to agree with the usual syntactic definition of a double negation translation due to Troelstra [23], provided  $\text{Th}(\mathcal{I}) = \text{Th}(\mathcal{C}) + (\text{DNE})$ , where  $\mathcal{I}$  comprises the involutive pocrimms in  $\mathcal{C}$ . See [2] for more information about the various syntactic double negation translations in  $\mathbf{AL}_{\mathbf{m}}$  and its extensions.*

**Theorem 5.2.1** *The Kolmogorov semantics,  $\mu^K$ , the Gentzen semantics,  $\mu^{\text{Gen}}$ , and the Glivenko semantics,  $\mu^{\text{Gli}}$ , are double negation semantics for any class of hoops.*

**Proof:** (DNS1) and (DNS3) are clear for  $\mu = \mu^{\text{Gli}} = \delta^{\mathcal{L}} \circ \mu^{\text{S}}$ , since  $\delta_{\mathbf{H}}^{\mathcal{L}} = \text{id}(H)$  when  $\mathbf{H}$  is involutive and  $\delta^{\mathcal{L}} \circ \delta^{\mathcal{L}} = \delta^{\mathcal{L}}$ . Also (DNS2) holds for  $\mu = \mu^{\text{Gen}}$  in the class of hoops, since, if  $\mathbf{H}$  is a hoop, then  $\text{im}(\delta)$  is an involutive subhoop, and for any  $\alpha : \text{Var} \rightarrow \mathbf{H}$ , we have:

$$\mu_{\mathbf{H}}^{\text{Gen}}(\alpha) = (\mu_{\mathbf{H}}^{\text{S}} \circ \delta^{\text{Var}})(\alpha) = \mu_{\mathbf{H}}^{\text{S}}(\delta \circ \alpha) = \mu_{\text{im}(\delta)}^{\text{S}}(\delta \circ \alpha).$$

Now it is easy to see using Theorem 5.1.1, that if  $\mathbf{H}$  is a hoop, then we have:

$$\mu_{\mathbf{H}}^{\text{K}} = \mu_{\mathbf{H}}^{\text{Gen}} = \mu_{\mathbf{H}}^{\text{Gli}}$$

Hence (DNS1), (DNS2) and (DNS3) hold for any of the three translations in any class of hoops.  $\blacksquare$

**Lemma 5.2.2** *Any pocrim satisfies  $\delta(a \rightarrow b) \geq \delta(a) \rightarrow \delta(b)$ .*

**Proof:** It is easy to see that (\*) if  $x + y = 1$ , then  $x \geq \neg y$  and (\*\*) if  $x \geq y$ , then  $\neg x + y = 1$ . Combining (\*) and (\*\*), we have (\*\*\*) if  $x + y = 1$  then  $\delta(x) + y = 1$ . Hence:

$$\begin{aligned} a + (a \rightarrow b) &\geq b \\ a + \neg b + (a \rightarrow b) &= 1 & (**) \\ \delta(a) + \neg b + \delta(a \rightarrow b) &= 1 & 2 \times (***) \\ \delta(a) + \delta(a \rightarrow b) &\geq \delta(b) & (*) \\ \delta(a \rightarrow b) &\geq \delta(a) \rightarrow \delta(b) & \blacksquare \end{aligned}$$

**Theorem 5.2.3** *The Kolmogorov semantics,  $\mu^K$ , is a double negation semantics for  $\mathbf{AL}_1$ .*

**Proof:** (DNS1) and (DNS3) are easy to verify. For (DNS2), by Theorem 3.2.1 it is enough to prove that, if  $\mathbf{AL}_{\mathbf{C}}$  proves  $A$ , then, for any pocrim  $\mathbf{P}$  and any  $\alpha : \text{Var} \rightarrow P$ ,  $\mu_{\mathbf{P}}^{\text{K}}(\alpha)(A) = 0$ . So let  $\mathbf{P}$  and  $\alpha : \text{Var} \rightarrow P$ , be given. We prove this by induction on a proof of  $A$ . The induction has an inductive step corresponding to our single inference rule and a base case for each of the axiom schemata used to define  $\mathbf{AL}_{\mathbf{C}}$ .

*Modus ponens:* by the inductive hypothesis, we are given that  $\mathbf{AL}_{\mathbf{C}}$  proves  $B$  and  $B \multimap A$ . Let  $a = \mu_{\mathbf{P}}^{\text{K}}(\alpha)(A)$  and  $b = \mu_{\mathbf{P}}^{\text{K}}(\alpha)(B)$  and note that from

the definition of  $\mu^K$  this means  $a \in \text{im}(\delta)$ . We want to show that  $a = 0$ . By the inductive hypothesis  $b = 0$  and  $\delta(b \rightarrow a) = 0$ , but then as  $a \in \text{im}(\delta)$  and using Lemma 5.2.2, we have  $a = \delta(a) = \delta(b) \rightarrow \delta(a) = 0$ .

For the axiom schemata, we have to show that if  $A$  is an instance of one of the schemata, then the semantic value  $X = \mu_{\mathbf{P}}^K(\alpha)(A)$  of the instance is equal to 0. We will make frequent and tacit use of the facts that  $x + y \rightarrow z = x \rightarrow y \rightarrow z$ ,  $x \geq \delta(x)$ , that  $x \rightarrow y \geq \delta(x) \rightarrow \delta(y)$  and that, if  $x \in \text{im}(\delta)$ , then  $x = \delta(x)$ .

(Comp): In this case,  $X = \delta(\delta(a \rightarrow b) \rightarrow \delta(\delta(b \rightarrow c) \rightarrow \delta(a \rightarrow c)))$ , for some  $a$ ,  $b$  and  $c$ , and we have:

$$\begin{aligned} \delta(\delta(a \rightarrow b) \rightarrow \delta(\delta(b \rightarrow c) \rightarrow \delta(a \rightarrow c))) &\leq \delta((a \rightarrow b) \rightarrow \delta(b \rightarrow c) \rightarrow \delta(a \rightarrow c)) \\ &\leq \delta((a \rightarrow b) \rightarrow (b \rightarrow c) \rightarrow (a \rightarrow c)) \\ &\leq \delta(0) = 0 \end{aligned}$$

(Comm):  $X = \delta(\delta(a + b) \rightarrow \delta(b + a))$ , for some  $a$  and  $b$ , so  $X = \delta(\delta(a + b) \rightarrow \delta(a + b)) = \delta(0) = 0$ .

(Curry):  $X = \delta(Y \rightarrow Z)$  where  $Y = \delta(\delta(a + b) \rightarrow c)$  and  $Z = \delta(a \rightarrow \delta(b \rightarrow c))$ , for some  $a$ ,  $b$  and  $c$ , and it is enough to prove  $Y \geq Z$ . We have:

$$\begin{aligned} \delta(\delta(a + b) \rightarrow c) &\geq \delta(a + b \rightarrow c) && (\text{as } a + b \geq \delta(a + b)) \\ &= \delta(a \rightarrow b \rightarrow c) \\ &\geq \delta(a \rightarrow \delta(b \rightarrow c)) && (\text{as } b \rightarrow c \geq \delta(b \rightarrow c)). \end{aligned}$$

(Uncurry):  $X = \delta(Y \rightarrow Z)$  where  $Y = \delta(a \rightarrow \delta(b \rightarrow c))$  and  $Z = \delta(\delta(a + b) \rightarrow c)$ , for some  $a, b, c \in \text{im}(\delta)$ , and it is enough to prove  $Y \geq Z$ . We have:

$$\begin{aligned} \delta(a \rightarrow \delta(b \rightarrow c)) &\geq \delta(a \rightarrow \delta(b) \rightarrow \delta(c)) && (\text{Lemma 5.2.2}) \\ &= \delta(a \rightarrow b \rightarrow c) && (\text{as } b, c \in \text{im}(\delta)) \\ &= \delta(a + b \rightarrow c) \\ &\geq \delta(a + b) \rightarrow \delta(c) && (\text{Lemma 5.2.2}) \\ &= \delta(a + b) \rightarrow c && (\text{as } c \in \text{im}(\delta)) \\ &\geq \delta(\delta(a + b) \rightarrow c). \end{aligned}$$

(Wk):  $X = \delta(\delta(a + b) \rightarrow a)$  where  $a \in \text{im}(\delta)$ . We have:

$$\begin{aligned}\delta(\delta(a + b) \rightarrow a) &= \delta(\delta(a + b) \rightarrow \delta(a)) \\ &\leq \delta((a + b) \rightarrow a) = \delta(0) = 0\end{aligned}$$

(EFQ): For some  $a$ ,  $X = \delta(1 \rightarrow a)$ , so  $X = \delta(0) = 0$ .

(DNE): For some  $a \in \text{im}(\delta)$ ,  $X = \delta(\delta(a) \rightarrow a) = \delta(a \rightarrow a) = \delta(0) = 0$ .  $\blacksquare$

**Example 5.2.1** Consider the pocrim  $\mathbf{Q}_6$  with six elements  $0 < p < q < r < s < 1$  and with  $+$ ,  $\rightarrow$  and  $\delta$  as shown in the following tables:

$+$	0	$p$	$q$	$r$	$s$	1	$\rightarrow$	0	$p$	$q$	$r$	$s$	1	$\delta$
0	0	$p$	$q$	$r$	$s$	1	0	0	$p$	$q$	$r$	$s$	1	0
$p$	$p$	$p$	$r$	$r$	$s$	1	$p$	0	0	$q$	$q$	$s$	1	$p$
$q$	$q$	$r$	$r$	$r$	1	1	$q$	0	0	0	$p$	$s$	$s$	$q$
$r$	$r$	$r$	$r$	$r$	1	1	$r$	0	0	0	0	$s$	$s$	$r$
$s$	$s$	$s$	1	1	1	1	$s$	0	0	0	0	0	$q$	$s$
1	1	1	1	1	1	1	1	0	0	0	0	0	0	1

$\mathbf{Q}_6$  is not involutive, as  $\delta(x) = x$  fails for  $x \in \{p, r\}$ . In  $\mathbf{Q}_6$ , double negation is an implicative homomorphism:  $\neg\neg x \rightarrow \neg\neg y = \neg\neg(x \rightarrow y)$  for all  $x, y$ . Double negation is not quite an additive homomorphism in  $\mathbf{Q}_6$ :  $\neg\neg x + \neg\neg y = \neg\neg(x + y)$  unless  $\{x, y\} \subseteq \{q, r\}$ , in which case  $\neg\neg x + \neg\neg y = r > q = \neg\neg(x + y)$ . As indicated by the block decomposition of the operation tables, there is a homomorphism  $h : \mathbf{Q}_6 \rightarrow \mathbf{Q}_4$ , where  $\mathbf{Q}_4$  is as discussed in Example 2.2.6. The kernel congruence of  $h$  has equivalence classes  $\{0, p\}$ ,  $\{q, r\}$ ,  $\{s\}$  and  $\{1\}$  which are mapped by  $h$  to 0,  $u$ ,  $v$ , 1 respectively in  $\mathbf{Q}_4$ .

**Theorem 5.2.4** (i) The Gentzen semantics  $\mu^{\text{Gen}}$  is not a double negation semantics for any class of pocrim that contains the pocrim  $\mathbf{Q}_6$  of Example 5.2.1. (ii) The Glivenko semantics  $\mu^{\text{Gli}}$  is not a double negation semantics for any class of pocrim that contains the pocrim  $\mathbf{P}_4$  of Example 2.2.6.

**Proof:** (i): We show that (DNS2) does not hold for  $\mu^{\text{Gen}}$  in  $\mathbf{Q}_6$ . Let  $V, W \in \text{Var}$  and let  $A$  be the formula  $(V \otimes W)^{\perp\perp} \multimap (V \otimes W)$ .  $A$  is an instance of (DNE) and so, by Theorem 3.2.1,  $\mu_{\mathbf{P}}^{\text{S}}(\alpha)(A) = 0$ , for any involutive pocrim  $\mathbf{P}$  and any  $\alpha : \text{Var} \rightarrow P$ . Thus (DNS2) requires  $\mu_{\mathbf{Q}_6}^{\text{Gen}}(\alpha)(A) = 0$  for any  $\alpha : \text{Var} \rightarrow Q_6$ . However, if  $\alpha(V) = \alpha(W) = r$ , we have:

$$\begin{aligned}\mu_{\mathbf{Q}_6}^{\text{Gen}}(\alpha)(A) &= \delta(\delta(r) + \delta(r)) \rightarrow \delta(r) + \delta(r) \\ &= \delta(q + q) \rightarrow q + q \\ &= \delta(r) \rightarrow r = q \rightarrow r = s \neq 0.\end{aligned}$$



(ii): we argue as in the proof of (A), but taking  $A$  to be  $V^{\perp\perp} \multimap V$ . Then, if  $\alpha(V) = q$ , we have:

$$\begin{aligned}\mu_{\mathbf{P}}^{\text{Gli}}(\alpha)(A) &= \delta(\delta(q) \rightarrow q) \\ &= \delta(p \rightarrow q) = \delta(p) = p \neq 0.\end{aligned}\quad \blacksquare$$

**Theorem 5.2.5** *Let  $\mathcal{C}_1$  comprise the two pocrimms  $\mathbf{P}_4$  and  $\mathbf{L}_3$  of Examples 2.2.3 and 2.2.6 and let  $\mathcal{C}_2$  comprise the two pocrimms  $\mathbf{Q}_6$  and  $\mathbf{Q}_4$  of Examples 5.2.1 and 2.2.6. Then:*

- (i) *The Gentzen semantics,  $\mu^{\text{Gen}}$ , is a double negation semantics for  $\mathcal{C}_1$ , but the Glivenko semantics,  $\mu^{\text{Gli}}$ , is not.*
- (ii) *The Glivenko semantics,  $\mu^{\text{Gli}}$ , is a double negation semantics for  $\mathcal{C}_2$ , but the Gentzen semantics,  $\mu^{\text{Gen}}$ , is not.*

**Proof:** (i): By Theorem 5.2.4,  $\mu^{\text{Gli}}$  is not a double negation semantics for  $\mathcal{C}_1$ . As for  $\mu^{\text{Gen}}$ , (DNS1) is easily verified. For (DNS3) and (DNS2), note that for any  $\alpha : \text{Var} \rightarrow P_4$ , we have:

$$\mu_{\mathbf{P}_4}^{\text{Gen}}(\alpha) = (\mu_{\mathbf{P}_4}^{\text{S}} \circ \delta^{\text{Var}})(\alpha) = \mu_{\mathbf{P}_4}^{\text{S}}(\delta \circ \alpha) = \mu_{\mathbf{L}_3}^{\text{S}}(\delta \circ \alpha)$$

where in the last expression we have identified  $\mathbf{L}_3$  with the subpocrim of  $\mathbf{P}_4$  whose universe is  $\text{im}(\delta)$ . Thus evaluation under  $\mu^{\text{Gen}}$  with an assignment in any pocrim in  $\mathcal{C}_1$  is equivalent to evaluation under the standard semantics,  $\mu^{\text{S}}$ , with an assignment in the involutive pocrim  $\mathbf{L}_3$ . (DNS3) and (DNS2) follow immediately from this.

(ii): By Theorem 5.2.4,  $\mu^{\text{Gen}}$  is not a double negation semantics for  $\mathcal{C}_2$ . As for  $\mu^{\text{Gli}}$ , (DNS1) and (DNS3) are immediate from the definition of  $\mu^{\text{Gli}}$ . For (DNS2), let  $A$  be a formula, such that  $\mu_{\mathbf{Q}_4}^{\text{S}}(\alpha)(A) = 0$ , for any assignment  $\alpha : \text{Var} \rightarrow \mathbf{Q}_4$ . As  $\mathbf{Q}_4$  is the only involutive pocrim in  $\mathcal{C}_2$ , we must show that  $\mu_{\mathbf{P}}^{\text{Gli}}(\alpha)(A) = 0$  for  $\mathbf{P} \in \mathcal{C}_2$  under any assignment  $\alpha : \text{Var} \rightarrow P$ . This is easy to see for  $\mathbf{P} = \mathbf{Q}_4$ , since the Glivenko semantics is the double negation of the standard semantics and  $\mathbf{Q}_4$  is involutive. As for  $\mathbf{P} = \mathbf{Q}_6$ , let  $\alpha : \text{Var} \rightarrow \mathbf{Q}_6$  be given. As discussed in Example 5.2.1, there is a homomorphism  $h : \mathbf{Q}_6 \rightarrow \mathbf{Q}_4$ , so, as  $\mu^{\text{S}}$  is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc}\text{Ass}(\mathbf{Q}_6) & \xrightarrow{\text{Ass}(h)} & \text{Ass}(\mathbf{Q}_4) \\ \downarrow \mu_{\mathbf{Q}_6}^{\text{S}} & & \downarrow \mu_{\mathbf{Q}_4}^{\text{S}} \\ \text{Sem}(\mathbf{Q}_6) & \xrightarrow{\text{Sem}(h)} & \text{Sem}(\mathbf{Q}_4)\end{array}$$

Hence, by the assumption on  $A$ , we have:

$$(h \circ \mu_{\mathbf{Q}_6}^S(\alpha))(A) = \mu_{\mathbf{Q}_4}^S(h \circ \alpha)(A) = 0$$

So  $\mu_{\mathbf{Q}_6}^S(\alpha)(A) \in h^{-1}(0) = \{0, p\}$ . As  $\delta(0) = \delta(p) = 0$ , we can conclude:

$$\mu_{\mathbf{Q}_6}^{\text{Gli}}(\alpha)(A) = \delta(\mu_{\mathbf{Q}_6}^S(\alpha)(A)) = 0. \quad \blacksquare$$

**Remark 5.2.2** Taken with the following lemma and Remark 5.2.1, Theorem 5.2.5 implies the existence of logics extending  $\mathbf{AL}_1$  in which the syntactic Gentzen translation meets Troelstra's requirements on a double negation translation but the syntactic Glivenko translation does not and vice versa.

**Lemma 5.2.6** If  $\mathbf{L}_3$ ,  $\mathbf{P}_4$ ,  $\mathbf{Q}_4$  and  $\mathbf{Q}_6$  are as in Theorem 5.2.5, then:

$$\begin{aligned} \text{Th}(\mathbf{L}_3) &= \text{Th}(\mathbf{P}_4) + (\text{DNE}) \\ \text{Th}(\mathbf{Q}_4) &= \text{Th}(\mathbf{Q}_6) + (\text{DNE}). \end{aligned}$$

**Proof:** For the first equation, the right-to-left inclusion holds because identities are preserved in subalgebras. For left-to-right, let us write  $\mathbf{P} \models A$  to mean  $\mu_{\mathbf{P}}^S(\alpha)(A) = 0$  for every  $\alpha : \text{Var} \rightarrow P$ . Assume  $\mathbf{L}_3 \models A$  and let  $W_1, \dots, W_k$  be the variables occurring in  $A$ . Define  $B$  to be  $(W_1^{\perp\perp} \multimap W_1) + \dots + (W_k^{\perp\perp} \multimap W_k)$ . We claim that  $\mu_{\mathbf{P}_4}^S(\alpha)(B \otimes B \multimap A) = 0$  for every  $\alpha : \text{Var} \rightarrow P_4$ , so that as  $\mathbf{AL}_m + (\text{DNE})$  proves  $B$ ,  $\text{Th}(\mathbf{P}_4) + (\text{DNE})$  proves  $A$ . To see this let an assignment  $\alpha : \text{Var} \rightarrow P_4$  be given. Then either (i)  $\text{im}(\alpha) \subseteq \{0, p, 1\}$ , in which case  $\mu_{\mathbf{P}_4}^S(\alpha)(A) = 0$ , since  $\alpha$  is an assignment into a subpocrim isomorphic to  $\mathbf{L}_3$  and  $\mathbf{L}_3 \models A$  by assumption, or (ii)  $\alpha(W_i) = q$  for some  $i$ , but then  $\mu_{\mathbf{P}_4}^S(\alpha)(W_i^{\perp\perp} \multimap W_i) = q$  and so  $\mu_{\mathbf{P}_4}^S(\alpha)(B \otimes B) \geq q + q = 1$ . In both cases, we have that  $\mu_{\mathbf{P}_4}^S(\alpha)(B \otimes B \multimap A) = 0$ , proving the claim. The proof of the second equation is similar using the facts that identities are preserved in quotient algebras and that, if  $\mathbf{Q}_4 \models A$  and  $\alpha : \text{Var} \rightarrow Q_6$ , then  $\mu_{\mathbf{Q}_6}^S(\alpha)(A) \in \{0, p\}$ , implying that  $\mathbf{Q}_6 \models (A^{\perp\perp} \multimap A) \multimap A$ .  $\blacksquare$

## 6 Concluding Remarks

The axiom  $A \otimes (A \multimap B) \multimap B \otimes (B \multimap A)$ , that we call (CWC), characterizes what seems to us to be an important landmark between affine logic, in which using an assumption destroys it, and standard logic, in which we may use an assumption as many times as we please. The importance of

this axiom is reflected algebraically in the rich properties enjoyed by hoops, the algebraic models of (CWC), when compared with the algebraic models of general affine logic, namely pocrim. However, many of these properties depend on algebraic laws whose derivations involve extremely intricate applications of the identity  $x + (x \rightarrow y) = y + (y \rightarrow x)$  that corresponds to the axiom (CWC). The methods of the present paper mitigate the problem of finding these derivations in many cases of interest.

Our original interest in Łukasiewicz logic arose from work on *continuous logic* [3]. In [1] we investigate the natural algebraic models for continuous logic and an intuitionistic analogue. These models comprise specialisations of hoops that we call *coops* which admit a halving operator  $x \mapsto x/2$  satisfying the law  $x/2 = x/2 \rightarrow x$ . There is a characterization of subdirectly irreducible coops very like Blok and Ferreirim’s result for hoops and the method for proving identities of the present paper carries over straightforwardly. Using it, one may prove, for example, the following “De Morgan” identity:  $\neg(x/2) = 1/2 + (\neg x)/2$ .

Bova and Montagna have shown that the quasi-equational theory of commutative GBL-algebras is PSPACE-complete and conjecture that the equational theory is also PSPACE-complete. Our indirect method of proof provides a heuristic that proves to be very successful on simple formulas with just a few variables. One might conjecture that the decision problem for a fixed number of variables admits a more tractable decision procedure. A problem would be to give a tractable description of the structure of the free hoop on  $n$  generators. Berman and Blok [4] have studied free  $k$ -potent hoops, but the assumption of  $k$ -potency is quite a strong one: e.g., a coop  $\mathbf{C}$  is  $k$ -potent iff  $C = \{0\}$ .

Semantic methods of some sort are the only way of obtaining results such as Theorem 5.2.5 that delimit the applicability of a given syntactic translation. Hyland [17] gives a semantic account of double negation translations in categorical terms. It would be interesting to attempt to integrate the categorical approach with the algebraic approach of the present paper.

**Acknowledgments** We thank: Franco Montagna for drawing our attention to [7] and for a manuscript proof that the equational theory of commutative GBL-algebras is a conservative extension of that of hoops; George Metcalfe for encouraging remarks and for pointers to the literature; and Isabel Ferreirim for helpful correspondence about the theory of hoops.

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